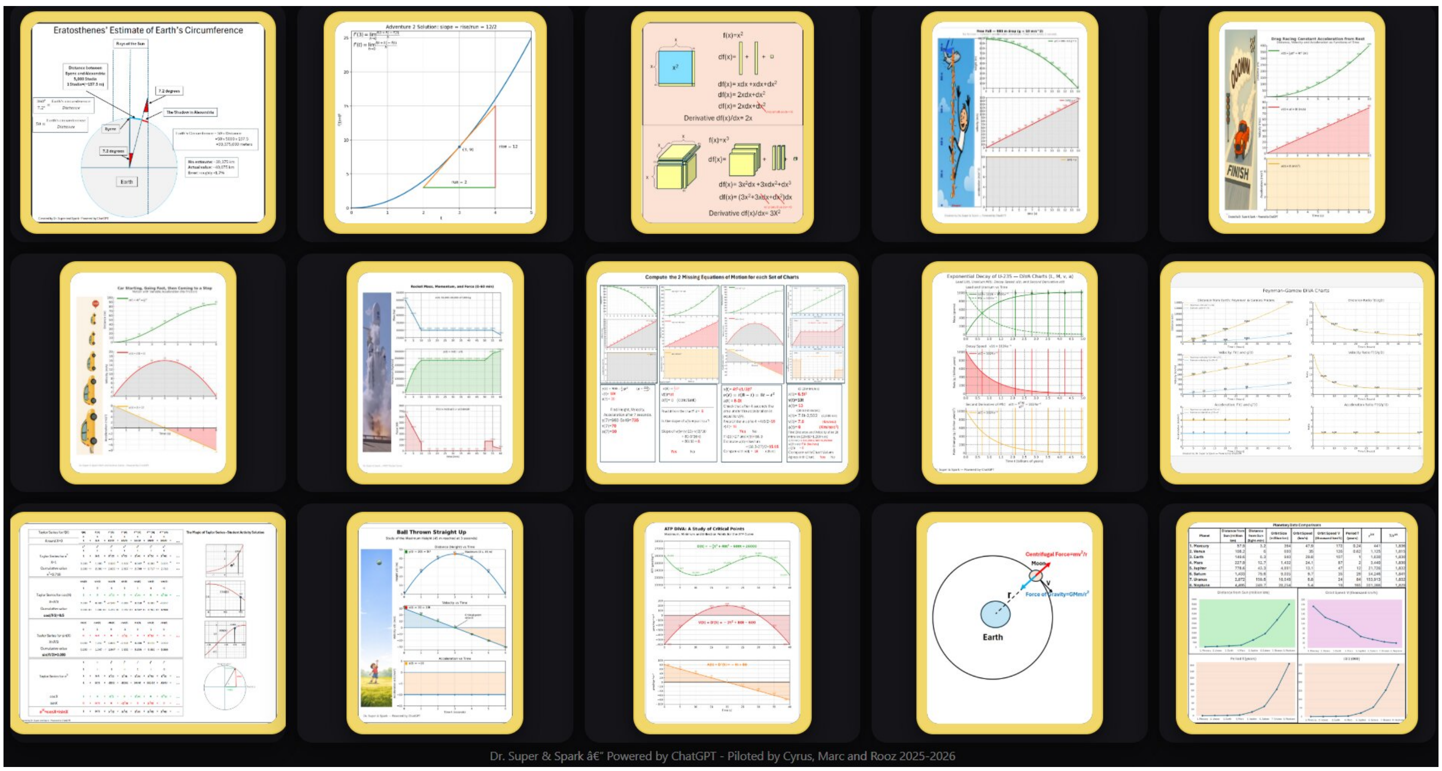
Adventures in Calculus & Mechanics for Young and Curious Minds



A Story-Powered Introduction to Calculus Through Motion, Graphs, and Discovery

Behrouz B. Aghevli, PhD

(Dr. Super)

First Edition

Tucson, Arizona

www.drsuper.com

ADVENTURES IN CALCULUS & MECHANICS

FOR YOUNG AND CURIOUS MINDS

A Story-Powered Introduction to Calculus Through Motion, Graphs, and Discovery

Behrouz B. Aghevli, PhD
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Based on the Wednesday Math Circles
Academic Year 2025–2026

Tucson, Arizona

with editorial and technical assistance from Spark (ChatGPT)

Dedication

I dedicate this book to my wonderful wife, **Shahine (Hedi)**, whose love, encouragement, and unwavering support have enriched every aspect of my life. She has always challenged me to remain active, curious, and intellectually engaged. These Adventures would never have come to life without her constant encouragement. It was also her insistence that mathematics should be connected to the physical world that brought **Mechanics** into our Math Circles and ultimately into this book.

I also dedicate this work to my grandchildren, **Cyrus and Rooz**, and to their friend **Marc**, whose enthusiasm, imagination, and endless questions transformed a collection of Math Circle lessons into a true adventure of discovery. Every activity in this book was tested, questioned, refined, and improved through our Wednesday explorations together. Their curiosity reminded me that mathematics is not merely something to be learned, it is something to be discovered.

May they always remain curious about the world and confident in their ability to explore it.

Acknowledgments

This book owes a great deal to the work and inspiration of many people.

First, I would like to express my deep appreciation to **Grant Sanderson (3Blue1Brown)**. His extraordinary *Essence of Calculus* series demonstrated that students could understand the fundamental ideas of calculus long before they master its techniques. His emphasis on intuition, visualization, and discovery helped shape the philosophy of these Adventures. One of the guiding principles of this book is that students should have the opportunity to experience the ideas of calculus for themselves—to see patterns, ask questions, make predictions, and, whenever possible, reinvent the mathematics rather than simply memorize procedures.

I would also like to thank **Bill Bryson**, whose wonderful book *A Short History of Nearly Everything* inspired many of the historical stories and scientific detours woven throughout these Adventures. Bryson's gift for explaining great discoveries through the lives of the people who made them helped remind me that mathematics and science are human stories of curiosity, perseverance, and imagination.

Special thanks are due to **Spark**, my ChatGPT assistant, for helping organize and edit my Math Circle notes and transform them into the Adventures presented here. Spark also assisted in developing the Teacher's Guide, the DiVA materials, and the hundreds of HTML pages, charts, illustrations, and supporting resources used in the online version of this project (drsuper.com).

Finally, I would like to thank the many teachers, parents, and students who believe that mathematics is best learned through exploration, discussion, and discovery. It is my hope that these Adventures help a new generation of young learners experience the same joy of mathematical discovery that has inspired mathematicians and scientists throughout history.

I also hope that teachers, parents, homeschoolers, and Math Circle leaders will use these Adventures as a starting point for their own journeys of exploration with young learners. If these stories, activities, and discussions spark curiosity, encourage questions, inspire deeper thinking, and help even a few students see mathematics as a living, creative, and profoundly human endeavor, then this project will have achieved its purpose

Introduction for Teachers, Parents, and Math Circle Leaders

Adventures in Calculus and Mechanics for the Young and Curious Minds is a complete story and video-powered pathway into calculus thinking for gifted students in upper elementary, and all students in middle school, and early high school. It was designed for teachers, parents, homeschoolers, and Math Circle leaders who want to introduce the fundamental ideas of calculus long before students encounter formal derivatives and integrals in a traditional course. It will also be an excellent introduction to calculus using the language of mechanics for the first week of any calculus course in high school or college.

Calculus is often taught as a collection of rules and formulas. Yet its central ideas—**change, motion, accumulation, prediction, and rates of change**—are concepts that young students can understand intuitively when they are presented visually and connected to meaningful stories and real-world situations. This series was created to make those ideas accessible, engaging, and memorable.

The Adventures grew out of weekly **Math Circles** that began during the **2022–23 school year**. Over the following years, students explored a wide variety of mathematical topics through guided discovery, puzzles, games, stories, and discussion. During the **2025–26 school year**, the focus turned to **Calculus and Mechanics**, resulting in the sequence of Adventures collected in this volume.

The original participants were **Cyrus (11)** and **Rooz (9)**, Dr. Super's grandchildren, together with their friend **Marc (11)**. Meeting each Wednesday for an hour to an hour and half, they worked on every activity, worksheet, story, chart, and discussion prompt found in these pages. Their questions, observations, and discoveries helped shape the lessons and often revealed better ways to explain difficult ideas. A separate PDF includes copies of all the completed worksheets by Cyrus, Rooz and Marc.

A central feature of the series is **DiVA charts**—the three stacked **Distance–Velocity–Acceleration** charts. DiVA was developed during the Math Circle sessions after it became clear that students could understand change more naturally by reading graphs before working with formulas. The charts allow students to see how motion unfolds and to discover for themselves the relationships between distance, velocity, and acceleration. DiVA became the common thread running through every Adventure.

How to Use This Book

Each Adventure is designed to be taught as a complete lesson and includes:

- A short video recommendation.
- A story or historical introduction.
- Student activity sheets.
- DiVA charts and graphing activities.
- Discussion questions.
- Solutions and teacher notes.

The recommended lesson flow is:

1. Begin with the Video

Start by watching the recommended video together. Most Adventures use the outstanding **3Blue1Brown Essence of Calculus** series or other carefully chosen educational videos.

Before pressing play, ask students:

- What do you think this video will be about?
- What questions do you already have?

- What do you expect to learn?

Encourage students to watch actively rather than passively. Pause occasionally to discuss important ideas and invite predictions about what comes next.

2. Discuss the Main Ideas

After the video, spend a few minutes discussing what students noticed.

Possible questions include:

- What surprised you?
- What was confusing?
- What pattern did you see?
- How does this connect to something you already know?

The goal is not perfect understanding. The goal is curiosity and engagement.

3. Read the Story Together

Each Adventure begins with a short story, historical episode, or real-world situation mostly from Bill Bryson's wonderful book: "A Short History of nearly Everything".

You may:

- Read the story aloud.
- Have students take turns reading.
- Photocopy the story pages for individual use.
- Use the online version if available.

The stories provide context and help students connect mathematical ideas to people, history, science, and everyday life.

4. Distribute the Student Activity Sheets

Photocopy the activity sheets and corresponding DiVA charts that are at the end of each Adventure, before the lesson.

Students should work individually or in small groups to:

- Complete the Activity Sheets.
- Complete graphs.
- Answer questions.
- Explain their reasoning.
- Discuss their observations.

Whenever possible, allow students to struggle productively before providing hints or explanations.

5. Focus on Explanation Before Calculation

A key principle of Adventures is:

Explain first. Calculate second.

Ask students questions such as:

- How do you know?

- What do you see in the graph?
- What must happen next?
- Why does that answer make sense?

The emphasis should always be on reasoning and interpretation rather than simply obtaining numerical answers.

6. Use the DiVA Charts Frequently

The DiVA charts are the heart of the program.

Students should continually ask:

- What is happening to the distance graph?
- What does that tell us about velocity?
- What does the velocity graph tell us about acceleration?

Over time, students begin to see derivatives and integrals as natural visual relationships rather than mysterious formulas.

7. Review and Reflect

End each session with discussion.

Possible questions include:

- What was the most interesting idea today?
- What surprised you?
- What would happen if we changed the situation?
- What do you think we will study next?

These conversations often reveal deeper understanding than written answers alone.

A Final Note

This book is not intended to be a traditional calculus course. It is an invitation to explore the ideas that made calculus one of the greatest achievements in human thought.

Students are encouraged to observe, predict, question, explain, and discover. The goal is not merely to learn calculus techniques but to develop the habits of mind used by mathematicians, scientists, engineers, and curious thinkers everywhere.

We hope these Adventures bring the same excitement, wonder, and joy that emerged during our Wednesday Math Circles with Cyrus, Rooz, and Marc—and that they inspire a new generation of young explorers to discover the beauty of Calculus and Mechanics for themselves.

Math Circle Notes

The Adventures in Calculus & Mechanics series grew out of weekly Math Circles conducted during the 2025–2026 academic year. For readers interested in seeing how these Adventures evolved through discussion, exploration, and student questions, the Math Circle Notes for this series have been created as a PDF as an Appendix to this volume on drsuper.com. The notes include the original activity worksheets completed by Cyrus, Rooz, and Marc, together with the discussions that accompanied them.

While the Adventures have been edited and organized for classroom and home use, the notes provide additional insight into the conversations, discoveries, and occasional detours that helped shape them. Some notes from Math Circles conducted in previous years on a variety of mathematical topics are also available in draft form on the Dr. Super website.

Table of Contents

🧑‍🏫 Adventure 1 – Area Under a Line, Slope, & the Big Idea of Calculus

🎯 Purpose of This Activity

This introductory session gently immerses students in the two central ideas of calculus:

- Accumulation — understanding total change by looking at the *area under a graph*.
- Instantaneous rate of change — understanding how fast something is changing *right now* through *slope*.

These ideas might sound advanced, but in fact they arise naturally from simple experiences—watching something move, grow, or change. Today's activity shows students how to *see* those ideas long before any formal calculus is introduced.

"When something moves, grows, or changes, calculus is quietly at work beneath the surface. Today, we begin learning how to see that change."

This sets the emotional tone for the series — exactly in the spirit of Grant Sanderson's You Tube 3Blue1Brown: Essence of Calculus Series and stories from Bill Bryson's: A short History of nearly Everything.

🎬 Why We Begin with a Story

We open the session with the remarkable story of Eratosthenes, who measured the entire circumference of the Earth using nothing more than:

- a stick,
- a shadow,
- a tiny angle, and
- a bit of geometry and reasoning.

Eratosthenes' reasoning is a perfect gateway to calculus because he used the same mindset students will learn today:

A small piece of information can reveal something enormous.

He used a line (the shadow)

a slope (the angle),

and a simple geometric slice of a circle

to understand the whole Earth.

This mirrors exactly what we ask students to do with area and slope:

- take a small slice,
- observe what it tells you,

- and reason your way to the whole picture.

After hearing the story, students estimate Earth's circumference themselves using the activity sheet titled: **Your Estimate of Earths Circumference,** and the solution sheet following it titled: **Eratosthenes' Estimate of Earth's Circumference**.

🎥 Video to Show Before the Worksheet

3Blue1Brown — *Essence of Calculus*, Chapter 1

This short segment introduces:

- area as accumulated change
- slicing a region
- curves and lines as measurable shapes

Tell students: "This video is the reason we can understand calculus visually." Then transition the worksheets.

📈 Why We Use Straight Lines First

To connect Eratosthenes' method to calculus, the worksheets use only the simplest possible functions:

- $y = x$
- $y = x + 2$

These lines let students discover:

- Area under a line → *accumulated total*
- Slope of a line → *how fast it changes*

All without algebra, without formulas, and without any of the symbolic machinery that often intimidates students.

Instead, they see:

- a shaded triangle,
- a rising line,
- a steeper line,
- a tiny slope angle,
- or a bigger one—

and begin to understand change visually, just as Grant Sanderson's 3Blue1Brown videos show.

⭐ What This Activity Is Really Teaching

This session sets the foundation for the entire Calculus & Mechanics Series.

Students learn that:

- Area = "how much happened"
- Slope = "how fast it's happening right now"
- A small slice can tell a big story
- Change has a structure you can see

These insights prepare them for everything to come:

free fall, acceleration, motion charts, rockets, exponentials, and more.

📊 Charts & Worksheets Used in This Activity

⭐ Worksheet 1 — Area Under the Line $y = x$

Students calculate triangle areas under the line exactly as shown on the problem set:

- Example: from $x = 0$ to $x = 3$
- General formula: $A = r^2/2$

All diagrams and step-by-step reasoning appear in the student worksheet and their solution sheets.

⭐ Worksheet 2 — Area Under the Line $y = x + 2$

Students follow the decomposition shown in the worksheet:

- **Rectangle**: area = $2x$
- **Triangle**: area = $x^2/2$
- **Total**: $A(x) = \frac{x^2}{2} + 2x$

The worksheet then introduces the *area curve* and its slope at $x = 2$.

Students estimate the tangent slope using the diagram on the sheet; the solution is given on the corresponding solution sheet.

The discovery is simple:

The slope of the area curve equals the value of the original line

(Our first encounter with the Fundamental Theorem of Calculus.)

🧠 Teacher Background

1. Areas Under Straight Lines

For $y = x$: area to r is $r^2/2$

For $y = x + 2$: area is $A(x) = x^2/2 + 2x$

2. Slope of the Area Curve

Since

$$A(x) = x^2/2 + 2x,$$

its derivative is

$$A'(x) = x + 2,$$

which matches the original line.

Derivative of accumulated area = original function.

This relationship is illustrated directly on the student worksheet.

🔍 Guiding Students Through the Activity

Part 1 — Triangle Under $y = x$

Ask:

“What shape do you see?”

“How do we find its area?”

“What changes when the triangle grows?”

Use the diagram on student activity sheet for visual support.

Part 2 — Trapezoid Under $y = x + 2$

Guide students to split the region (as shown on their sheet):

rectangle

triangle

total area

Encourage observations:

“The line is higher, so the area increases more quickly.”

“Adding 2 shifts everything upward.”

Part 3 — Slope at $x = 2$

Using the tangent diagram on Page 2 of the student activity sheets:

Ask:

“How steep is the curve at this point?”

“Where do we see this number on the line $y = x + 2$?”

Students will see that the slope equals the line’s height at $x = 2$.

💬 Discussion Questions

“Why does the area grow faster under $y = x + 2$?”

“What does slope tell us about change?”

“How does a steeper line change the area?”

"What does the tangent line show?"

These questions naturally lead into Section 1 (motion).

🧩 Extensions

Draw your own line → shade area → find formula

Compare growth of $y = x$vs $y = x + 2$

Predict the area formula under $y = x + 5$

Students extend patterns instinctively.

🧪 Teacher Tips

Focus on **visual reasoning**, not arithmetic

Use hand gestures for slope and steepness

Ask: "What do you **see** happening?"

Connect area ↔ "how much happened"

Connect slope ↔ "how fast it's happening right now"

🧱 Closing Notes

This activity introduces the mindset behind all future sections:

Area = accumulated change

Slope = instantaneous change

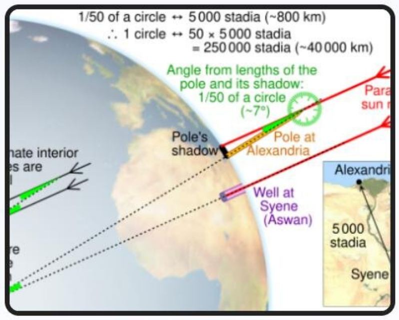
Straight lines reveal these ideas cleanly

Calculus becomes a way of *seeing* motion

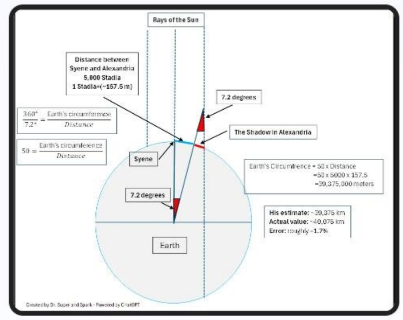
This prepares students for all the other exciting Adventures in Calculus and Mechanics Series that is grounded in **3Blue1Brown's visual clarity** and **Bill Bryson's sense of wonder**.

🌍 Adventure 1 Story – Measuring the World by Slices and Sticks

Eratosthenes teaching in Alexandria by Bernardo Strozzi (1635)



Measure of Earth's circumference, based on the approximation that Syene is on the Tropic of Cancer and on the same meridian as Alexandria



Eratosthenes' Estimate of Earth's Circumference Student Worksheet

Inspired by Bill Bryson's chapter on early attempts to measure Earth

Long before satellites and laser measuring tools, no one knew how big the Earth was. People guessed, argued, and made wildly different claims. Some thought the planet was enormous; others believed it was much smaller. What they all agreed on was this: the Earth was far too large to measure directly.

Then came Eratosthenes.

He lived more than two thousand years ago and worked in the Great Library of Alexandria — a place filled with scrolls, scholars, and a great deal of sunshine. Eratosthenes had learned something curious: in a southern city called Syene, the sun shone straight down a well on the summer solstice. At the very same moment in Alexandria, a vertical stick cast a small shadow.

This tiny difference — the angle of a single shadow — was the clue.

Eratosthenes realized he could "slice" the Earth using shadows the same way we slice area under a line: by taking a simple measurement and using it to understand something much bigger.

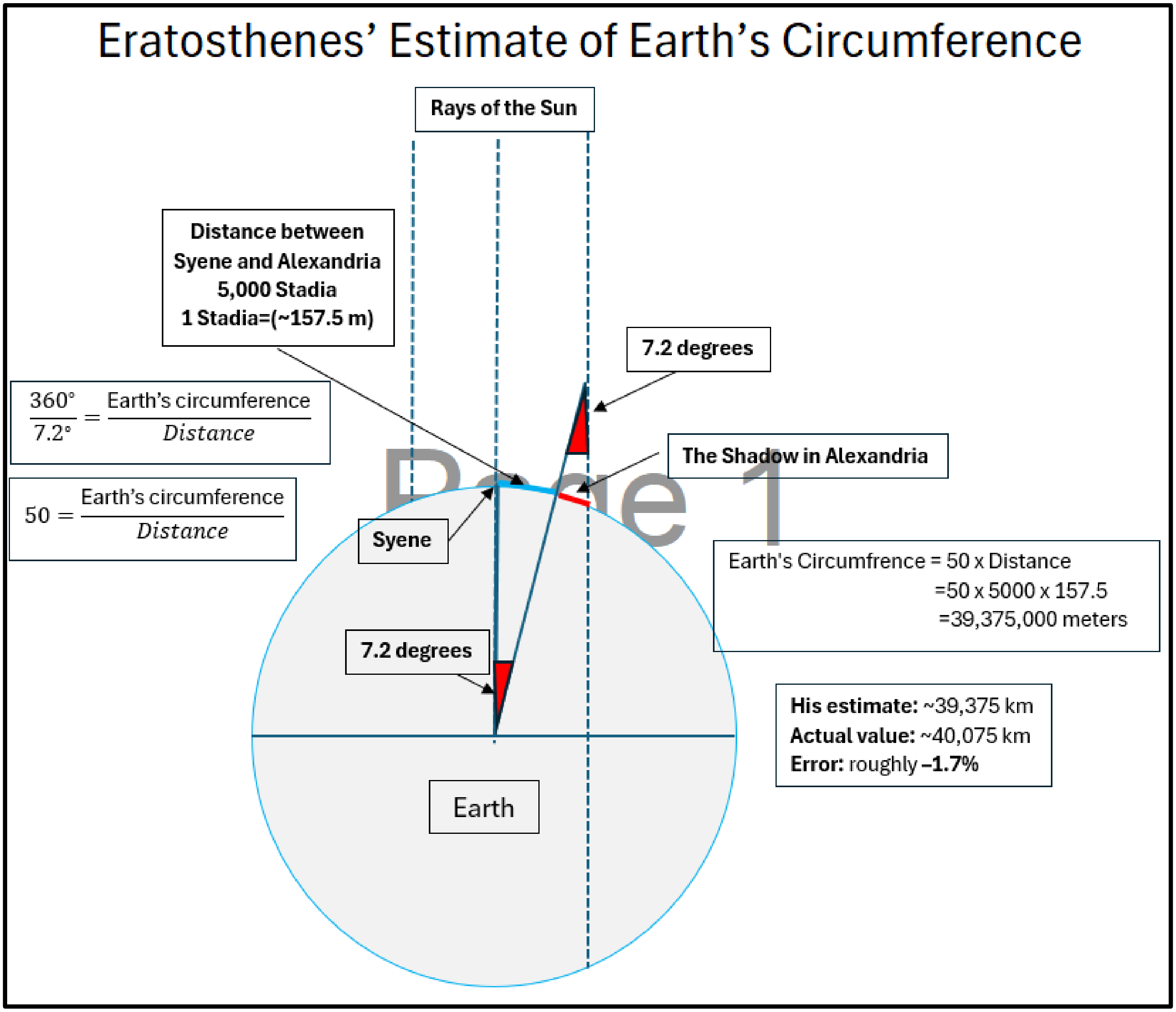
He measured the angle of the shadow in Alexandria (about 7 degrees), knew the distance between the two cities, and reasoned:

"If 7 degrees of curvature corresponds to this many miles, then the whole 360 degrees must correspond to... the entire Earth."

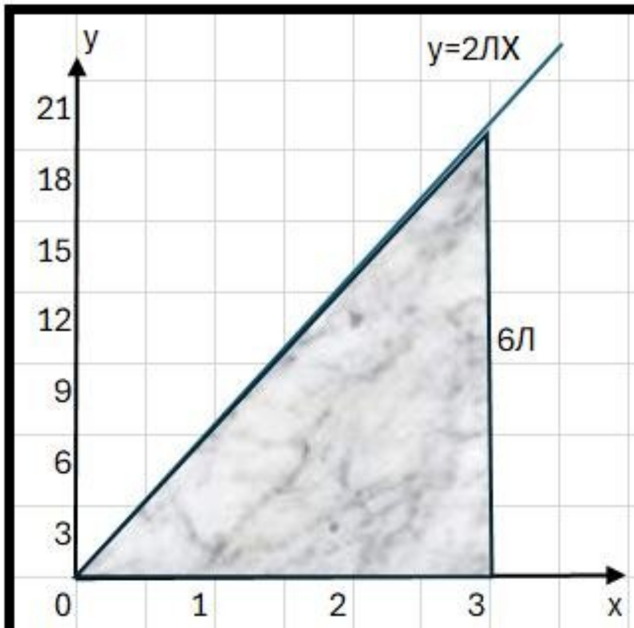
In other words, he used a small piece — one slice — to understand the whole curve of the planet.

His method was astonishingly accurate, less than 2% error as compared to modern measurements.

🌍 Measuring the World by Slices and Sticks – Student Worksheet Solution

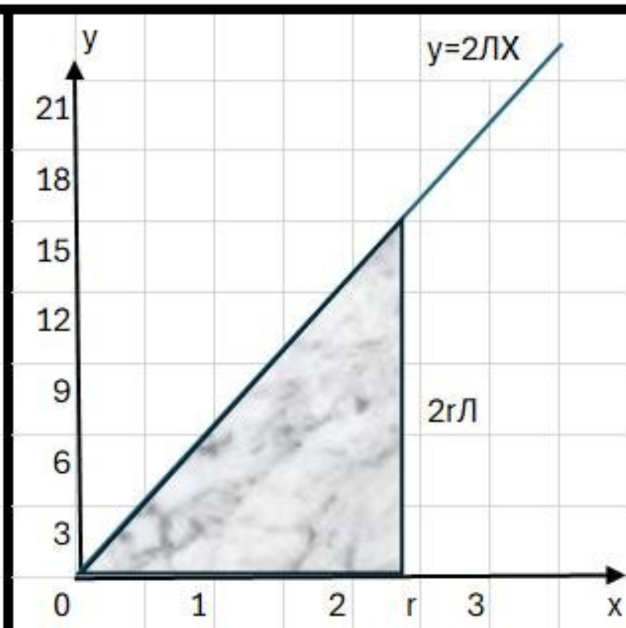


🌍 Area Under Line and Tangents – Student Worksheet Page 1- Solution



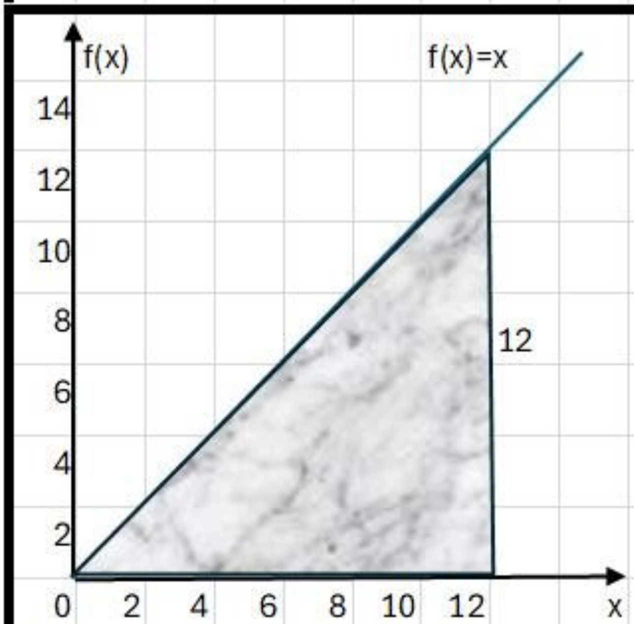
Find the area of the shaded Triangle:

Area = 3x6Л/2=9Л



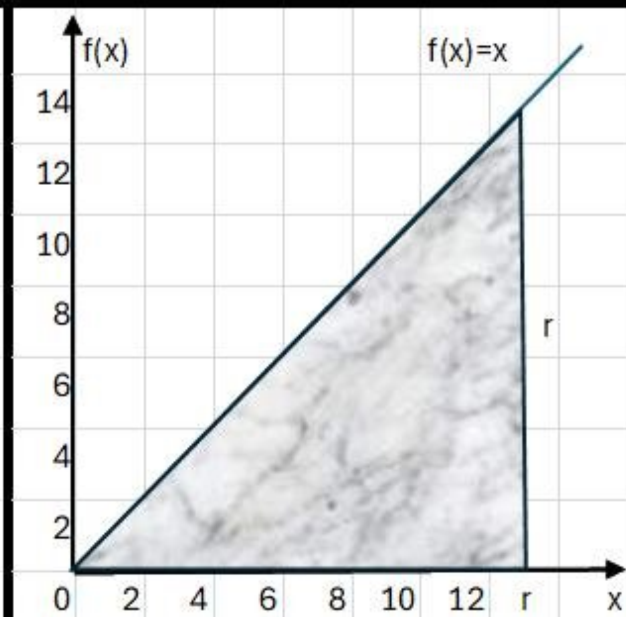
Find the area of the shaded triangle in terms of r

Area = 1/2(r)(2rЛ)=(Лr^2)



Find the area of the shaded Triangle:

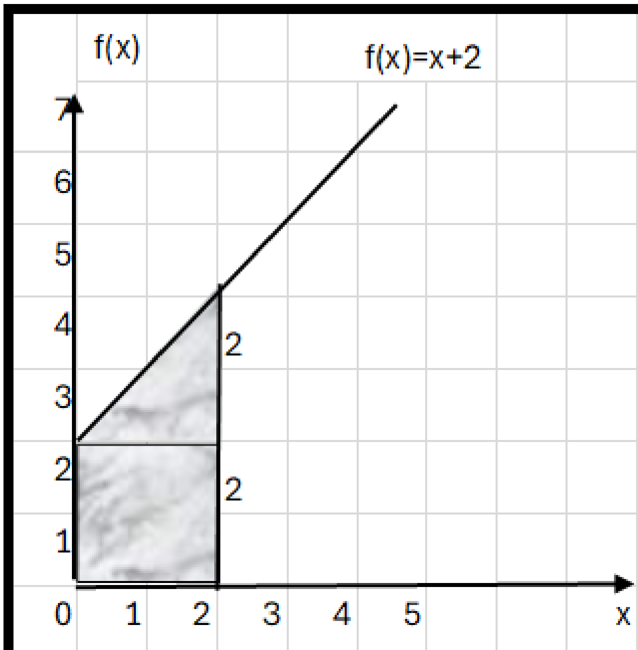
Area = 1/2(12x12)= 144/2=72



Find the area of the shaded triangle in terms of r

Area = 1/2(r)(r)=$r^2/2$

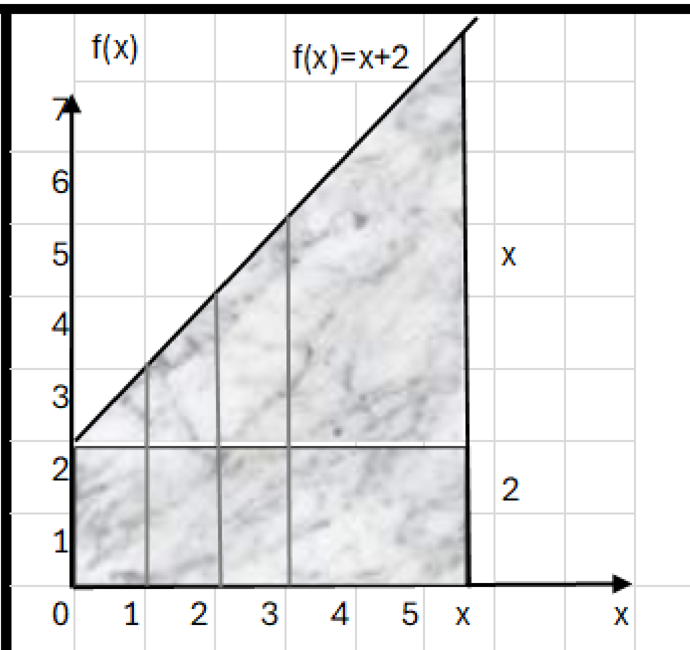
Area Under Line and Tangents – Student Worksheet Page 2- Solution



Find the area of the shaded Trapezoid:

Total Area = Area of triangle + Area of rectangle

= 2 + 4 = 6



Find the area of the shaded Trapezoid:

Total Area = Area of triangle + Area of rectangle

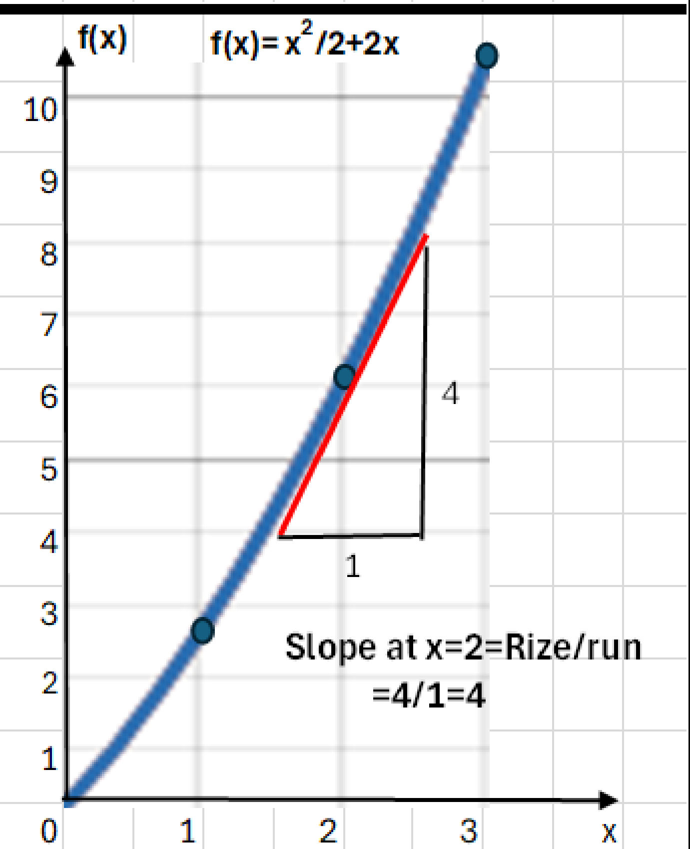
= $x^2/2$ + $2x$ = x^2+2x

x	Area of the Trapezoid at x
0	0
1	2.5
2	6
3	10.5

x	$f(x)=x^2/2+2x$
0	0
1	2.5
2	6
3	10.5

The area under the line x+2 at any point x is equal to the value of $x^2/2+2x$. We say $x^2/2+2x$ is the integral of x+2.

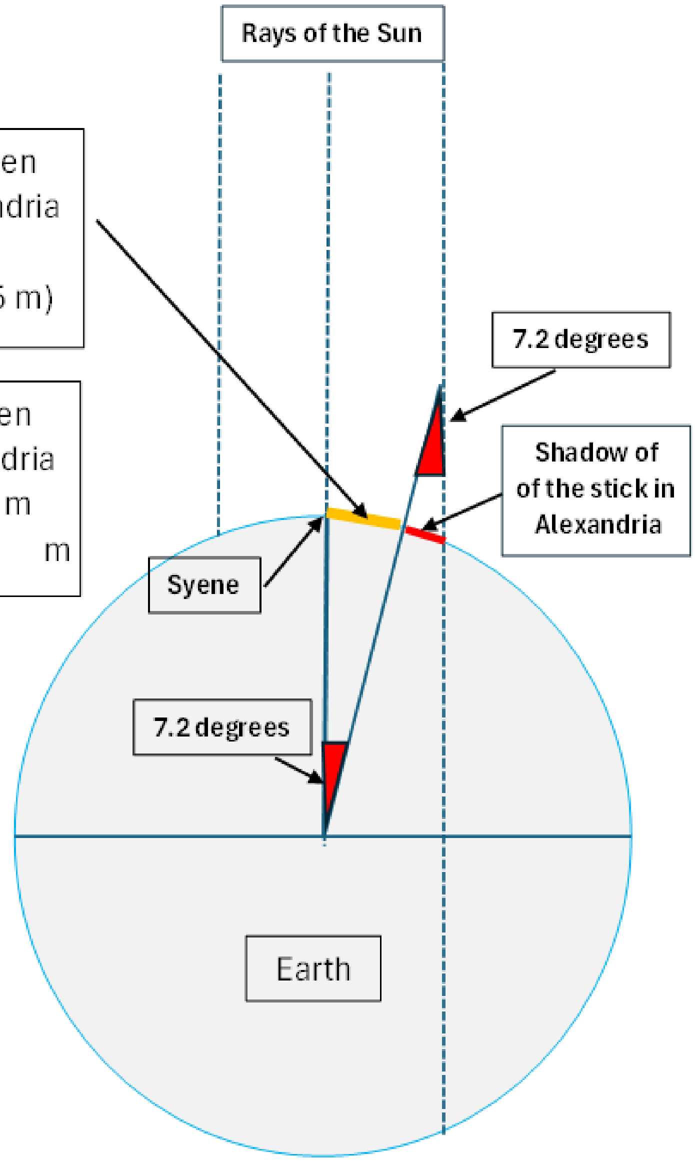
Also the slope of line tangent to $x^2/2+2x$ at point x=2 is 4 that is equal to x+2 we say x+2 is derivative of $x^2/2+2x$



Your Estimate of Earth's Circumference

Distance between Syene and Alexandria
5,000 Stadia
1 Stadia=(~157.5 m)

Distance between Syene and Alexandria
= 5,000x157.5 m
D = m



$R = \frac{360^\circ}{7.2^\circ} = \frac{\text{Earth's circumference}}{D}$

$R = \frac{\text{Earth's circumference}}{D} =$

Earth's Circumfrence = R x D
= m
= x (1Km/1000m)= km

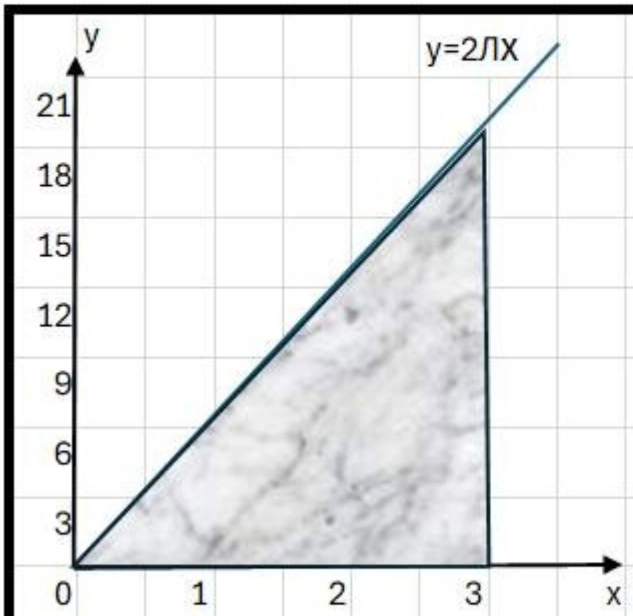
Your Estimate: km
Actual value: ~40,075 km

Error= Actual-Estimate/Actual

=

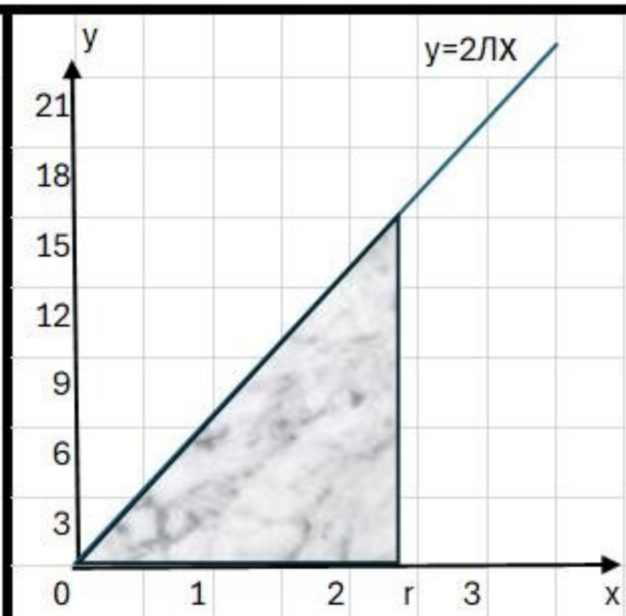
% Error= Error x 100 = %

🌍 Area Under Line and Tangents – Student Worksheet Page 1



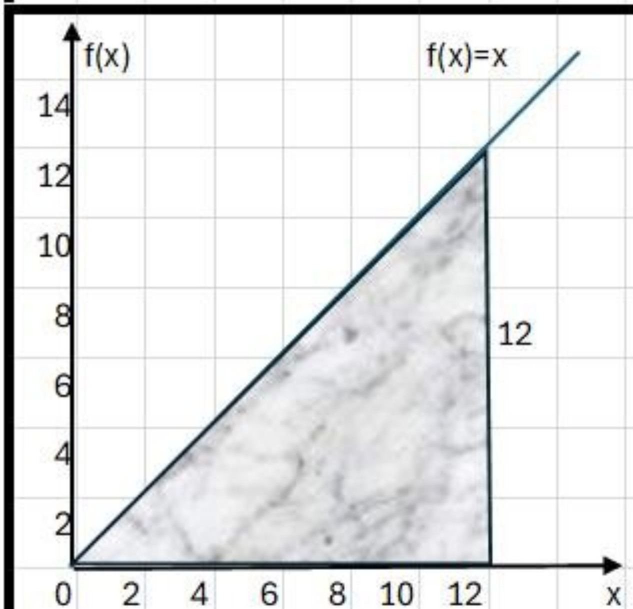
Find the area of the shaded Triangle:

Area =



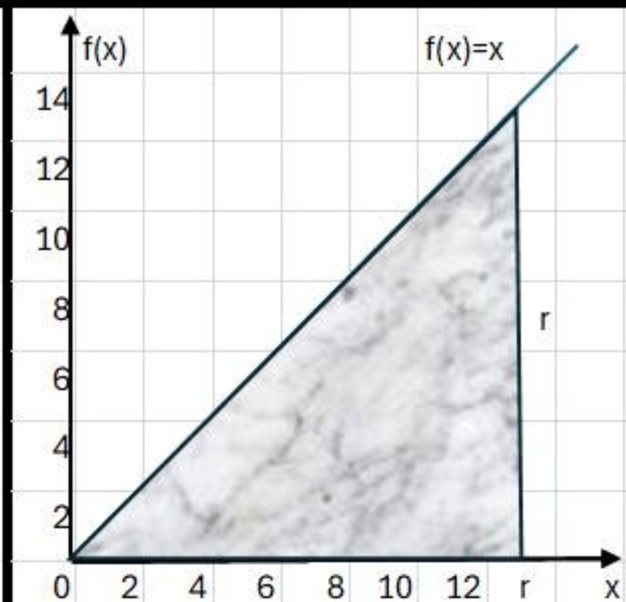
Find the area of the shaded triangle in terms of r

Area =



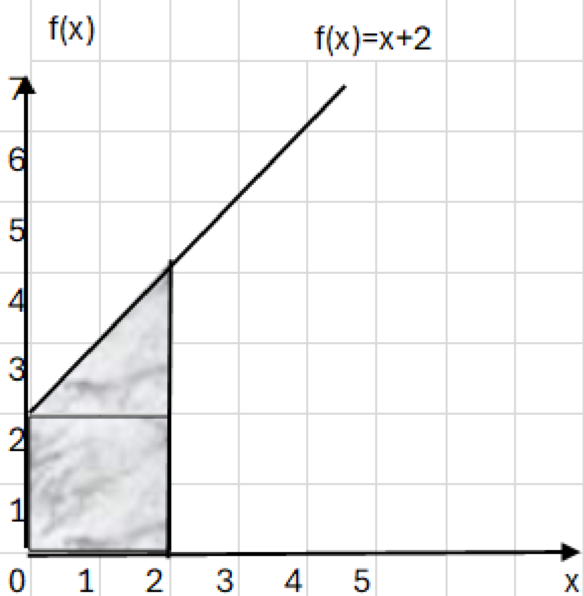
Find the area of the shaded Triangle:

Area =



Find the area of the shaded triangle in terms of r

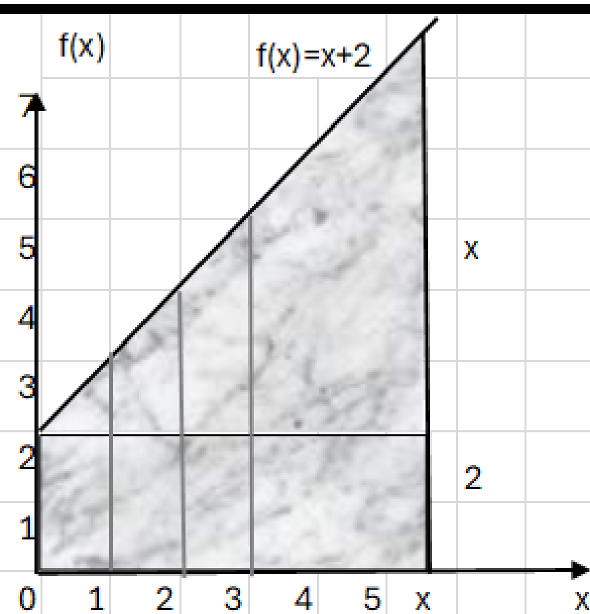
Area =



Find the area of the shaded Trapezoid:

Total Area = Area of [triangle] + Area of [rectangle]

= + =



Find the area of the shaded Trapezoid:

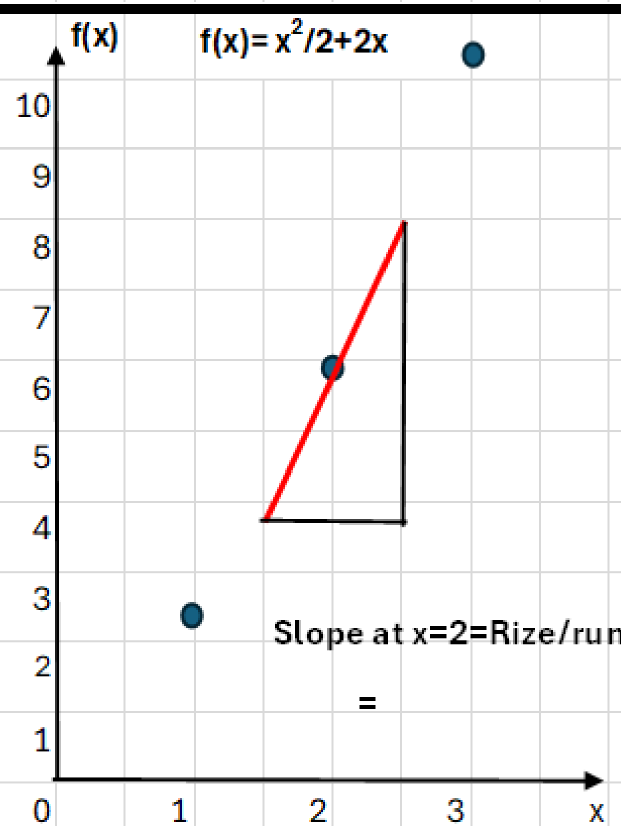
Total Area = Area of [triangle] + Area of [rectangle]

= + =

x	Area of the Trapezoid at x
0	
1	
2	
3	

x	$f(x)=x^2/2+2x$
0	
1	
2	
3	

Explain how the area under the line relates to the function $f(x)=x^2/2+2x$ and what is the slope of this function is related to



Slope at x=2=Rize/run

=

🧑‍🏫 Adventure 2 – Secret of Slope

🎯 Learning Objective

Students will develop a deep intuitive understanding that:

📈 A curve can have different steepness at different points

📏 A tangent line represents the steepness at a single instant

⚡ Slope describes instantaneous change

🧭 Curved motion can behave locally like straight motion

This Adventure introduces the core conceptual doorway to calculus.

🧠 Big Idea ✨ A tangent line tells the local story of a curve.

Students are not yet learning derivative rules.

They are learning how change behaves at one moment.

📖 Opening Story

Begin with the Newton apple narrative as a story of wonder, not a history lesson.

Emphasize:

🍎 The apple does not fall at constant speed

⏱️ Its motion changes continuously

📊 The graph of its motion is curved

📍 At each instant, the motion has a definite steepness

Teacher language suggestion:

"If time could pause, how fast is the change right at that instant?"

Avoid formulas and symbolic explanations.
The goal is curiosity about local behavior.

🎥 Video Segment

🎬 3Blue1Brown — *The Paradox of the Derivative (Chapter 2)*

Guide student attention to:

🎯 Slope stabilizing to a single value

💡 Instantaneous change becoming meaningful as limits

Pause and ask:

🤔 When does the slope the tangent to the curve keep changing?

🤔 What does this mean for motion?

✏️ Student Activity

Students are given:

📈 A second-degree curve

📍 A marked point on the curve

📏 A tangent line at that point

Students should:

📐 Measure rise over run

🔢 Compute the slope numerically

🗣️ Interpret the slope in words

Teacher emphasis:

"What does this slope tell us about how the motion is changing?"

The focus is interpretation, not symbolic manipulation.

🔍 Conceptual Discussion

Guide students toward the realization that:

📍 Slope is a local property

🔎 When we zoom in, curves appear straight

📏 A tangent line represents instantaneous behavior

Encourage questions such as:

❓ Would the slope be the same at another point?

❓ Why does the curve bend while the tangent remains straight?

❓ How can motion be changing yet still have a definite slope?

This builds:

✔ readiness for derivative concepts
✔ intuition for motion and mechanics
✔ understanding of local linear behavior

🌟 Conceptual Milestone

Students should leave with the understanding:

🌟 Even though change is continuous, at each instant it has a definite rate.

This is the birth of instantaneous thinking, the foundation of calculus.

🧩 Connection to Mechanics

Relate slope to physical ideas such as:

🚗 Changing speed

🪂 Acceleration intuition

📊 Motion graphs

Keep the discussion qualitative and intuitive.

Avoid introducing formal expressions such as $v = \frac{dx}{dt}$ at this stage.

🧭 Teacher Guidance Notes

Encourage:

🧠 Visual reasoning

🗣️ Verbal explanations

📐 Approximate slope reasoning

🎯 Conceptual clarity over algebraic precision

Avoid:

❌ Derivative rules
❌ Limit formalism
❌ Integral concepts
❌ Algebraic proofs

These will be introduced in later Adventures.

🧠 Closing Reflection (Teacher Script)

Conclude the session with:

🌌 "Today we learned that curves have a local story.
Calculus begins when we understand what happens at one instant."

This prepares students emotionally and intellectually for:

➡️ Difference quotient reasoning in the next Adventure.

📌 Teacher Outcome Target

If the Adventure is successful, students will be able to say:

💬 "Slope can describe motion at a single moment."

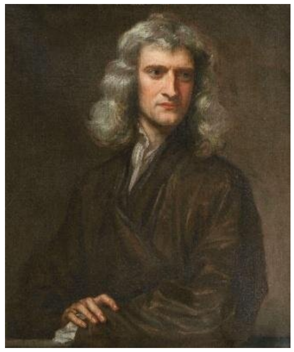
💬 "A tangent line shows instantaneous change."

💬 "Curves behave like lines when viewed locally."

If students can express these ideas, the conceptual goal has been achieved.

Slope

Adventure 2 Story – Newton and Leibniz – The Apple, the Area under a Curve and the Secret of Slope



Sir Isaac Newton **Gottfried Wilhelm Leibniz**

A long time ago — way before rockets, cars, or TikTok — people saw things move but didn't really understand *how* they moved. They knew rocks roll downhill. They knew apples fall. They knew rivers curve and shadows stretch. But nobody knew how to describe exactly **how fast** things were changing.

Then one afternoon in the 1660s, a young student named **Isaac Newton** sat under an apple tree to rest. He was home from school because of the plague, and he was tired from studying.

And yes — as the kids always ask — **an apple really did fall and hit him on the head.**

Newton rubbed his head and suddenly wondered: **"Why did the apple fall faster and faster as it got close to the ground?"** Why didn't it fall at one steady speed? That tiny question opened the door to one of the biggest ideas in math and science. Newton started drawing curves — little hills and loops — in his notebook. He noticed that if he drew a tiny straight line touching the curve at just one point, that line told him **how steep the curve was right there.** That little line is called a **tangent**. And the steepness of that line is called the **slope**.

Newton realized: **slope = the story of how something is changing at that moment.**

Meanwhile, in Germany, another brilliant thinker named **Gottfried Leibniz** was exploring a different big idea. Instead of looking at change, he studied **how things add up** — the area under a line, or how far something travels when its speed keeps changing. He sliced shapes into super-tiny strips, added them up, and invented the symbol: $\int dx$

which means "lots of little pieces added together.": Newton's idea was about **changes**. Leibniz's idea was about **accumulation**. Together, they became the two great sides of **calculus**.

Today, in Math Circle, you get to follow in their footsteps.

When you draw a tangent line at the point (3, 9), you are doing exactly what Newton started.

You're finding the slope — the secret of the curve.

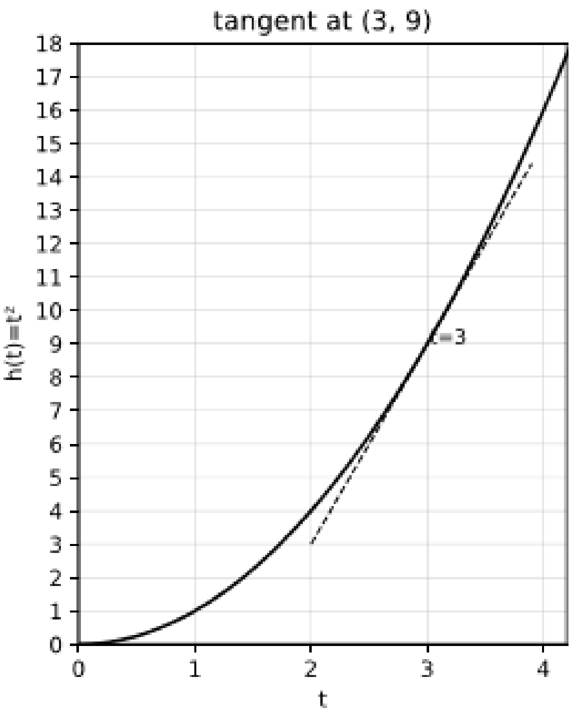
And when you shade the area under a line, you're doing what Leibniz did — discovering how little pieces add up to something big.

By the end of today, you'll know something amazing: Lines and Curves tell stories; Slopes tell how fast things change. Areas tell how things accumulate. And calculus is the language that helps you read it all.

🧑‍🔬 Student Activity Solution

Calculus Activity #1 — The Tangent Line and the Derivative

(Based on 3Blue1Brown — The Essence of Calculus (2): The Tangent Line)



$$\frac{dh(3)}{dt} = \frac{h(3+\Delta t) - h(3)}{\Delta t}$$

$$\frac{(3+\Delta t)^2 - 9}{\Delta t} = \frac{9 + 6\Delta t + (\Delta t)^2 - 9}{\Delta t} = 6 + \Delta t$$

$$\Delta t \to 0,\ \frac{dh(3)}{dt} = 6$$

$$\frac{dh(t)}{dt} = \frac{h(t+\Delta t) - h(t)}{\Delta t}$$

$$\frac{(t+\Delta t)^2 - t^2}{\Delta t} = \frac{2t\Delta t + (\Delta t)^2}{\Delta t} = 2t + \Delta t$$

$$\Delta t \to 0,\ \frac{dh}{dt} = 2t$$

t	$h(t)=t^2$
0	0
1	1
2	4
3	9
4	16

The slope (derivative) tells us how fast h(t) is changing at a moment.

At t = 3, slope = 6 means h(t) is increasing by 6 units per second right then —

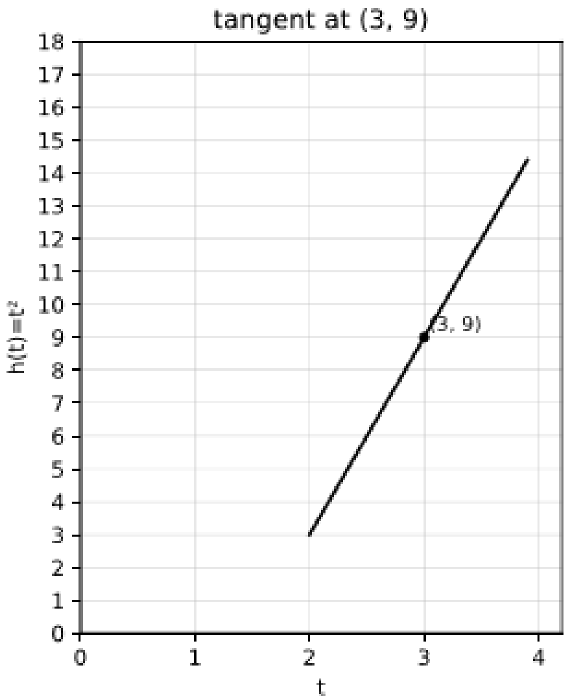
the instantaneous rate of change.

Dr. Super & Spark — Powered by ChatGPT | Based on 3Blue1Brown Calculus Series

Student Activity Sheet

Calculus Activity #1 — The Tangent Line and the Derivative

(Based on 3Blue1Brown — The Essence of Calculus (2): The Tangent Line)



$$\frac{dh(3)}{dt} = \frac{h(3+\Delta t) - h(3)}{\Delta t} =$$

$$\frac{(3+\Delta t)^2 - 9}{\Delta t} =$$

$$\Delta t \to 0, \ \frac{d(h(3))}{dt} =$$

$$\frac{dh(t)}{dt} = \frac{h(t+\Delta t) - h(t)}{\Delta t} =$$

$$\frac{(t+\Delta t)^2 - t^2}{\Delta t} =$$

$$\Delta t \to 0, \ \frac{dh(t)}{dt} =$$

t	$h(t)=t^2$
0	
1	
2	
3	
4	

---

1) Complete the table and draw this function.

2) What is the slope of the tangent to this function at point (3, 9) or any point (t, t^2)?

3) How is that related to what you found on top for t = 3 and any t in general?

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🧑‍🏫 Adventure 3 – The Power Rule Through Geometry: from Algebra to Calculus

🧩 1. Purpose of This Adventure

Adventure 3 is the first time students **see where the Power Rule comes from**.

The goal is not to memorize

$$\frac{d(x^2)}{dx} = 2x \text{ and } \frac{d(x^3)}{dx} = 3x^2$$

but to **experience them geometrically**.

Students should understand:

- A square grows in **two directions**.
- A cube grows in **three directions**.
- The derivative comes from counting the "new pieces" created by growth.
- Terms involving $dx^2$and dx^3become negligible as $dx \to 0$.

This is their first true structural encounter with calculus.

📖 2. Story Connection (Historical Framing)

Begin by reading the Adventure 3 Story: "Thousand-Year Journey from Algebra to the Power Rule"
Emphasize:

- Algebra was invented to create a universal mathematical language.
- Omar Khayyam pushed cubic equations beyond algebra alone.
- Italian algebraists discovered imaginary numbers while solving cubic equations.
- Calculus became possible only after algebra matured.
- Newton and Leibniz gave us the notation students now use.

Tell the students:

"Today, you are stepping into the moment where algebra turns into calculus."

The story should create intellectual excitement before the geometry begins.

▶ 🎥 3. Warm-Up (Video + Physical Setup)

Watch the 3Blue1Brown video **Derivative formulas through geometry | Chapter 3, Essence of calculus** . It shows how formulas similar to **$d(x^2)/dx = 2x$** can be seen as geometry.

Pause intentionally to:

- Highlight the idea of thin strips.
- Ask: "Where is the new area coming from?"

- Connect the animation to the printed worksheet diagrams

Adventure 3 Student Activity Sheets

If possible, use physical tiles or Algebra Lab Gear to model the strips.

The physical model helps students who struggle with abstraction.

🟩 4. Activity A — Growing the Square (x^2)

Distribute Adventure 3 Student Activity Sheet: ***Discovering the Power Rule by Watching Shapes Grow.*** And the

Key teaching moves:

1. Make students mark **dx clearly**.
2. Ask:
    - “Where does the new area appear?”
    - “Why are there two strips?”
3. Have them write:
    - $x \cdot dx$
    - $x \cdot dx$
    - dx^2

Then guide them to:

$$df = 2x\,dx + dx^2$$

Critical moment:
Ask:

“When dx is extremely small, which term dominates?”

Students must articulate why dx^2 becomes negligible.

Do not rush this.
This is the conceptual hinge of the lesson.

Conclude:

$$\frac{df}{dx} = 2x$$

Ask: “Why 2? Why not 1 or 3?”

Answer: because a square grows in **two directions**.

🟦 5. Activity B — Growing the Cube (x^3)

Refer to pages 3–4

Adventure 3 Student Activity Sheet

Students should identify:

- Three face sheets ($x^2 \cdot dx$ each)
- Three edge strips ($x \cdot dx^2$)
- One tiny cube (dx^3)

Guide them to:

$$df = 3x^2dx + 3xdx^2 + dx^3$$

Then factor:

Again ask:

"As dx → 0, what remains?"

Result:

$$\frac{df}{dx} = 3x^2$$

Then ask the key generalization question:

"If a shape grows in 4 independent directions, what would happen?"

Let them predict the pattern.

🔍 6. Discussion Prompts

Use these to deepen understanding:

- Why does the coefficient equal the number of growth directions?
- Why do higher powers of dx vanish?
- Why does this method depend on algebra existing first?
- Could Khayyam have written this derivative?
- Why did imaginary numbers matter historically?

🧠 7. Anticipated Difficulties

Students may:

- Confuse df with df/dx.
- Forget to factor out dx.
- Not understand why dx^2 disappears.
- Mechanically compute without conceptualizing growth.

If needed, return to physical strips.

⭐ 8. What Success Looks Like

By the end of Adventure 3, students should be able to say:

- “A square grows in two directions.”
- “A cube grows in three directions.”
- “The derivative measures how fast area or volume grows.”
- “The Power Rule is not magic — it comes from geometry.”

If they can explain it verbally without writing formulas, the adventure succeeded.

🧱 9. Advanced Extension Ideas (Optional)

- Ask students to predict derivative of x^4.
- Connect to derivative of 1/x using area of constant rectangle.
- Preview how this leads to motion (velocity & acceleration).
- Show how Bernoulli used the Power Rule in physics.

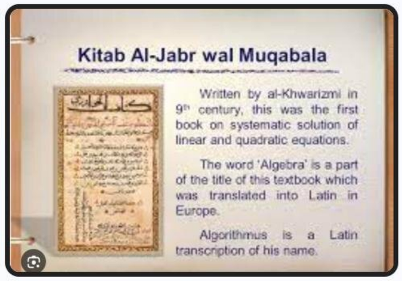
🪄 10. Closing Reflection

Remind students: “It took mathematicians over 1,000 years to build the language that allowed us to see this simple pattern.”

Adventure 3 marks the true turning point from algebra to calculus.

⭐ Adventure 3 Story – Thousand-Year Journey from Algebra to the Power Rule

Persian Polymath Mūsā al-Khwārizmī



al-Jabr wa'l-Muqābala, (Restoring and Balancing)

$$\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}$$
$$\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}$$

Cardano's Solution to the Third Degree Equation x3+qx+p

More than a thousand years ago, in the golden-age city of Baghdad, a quiet Persian scholar named al-Khwārizmī wandered through the long halls of the House of Wisdom, surrounded by scrolls from Persia, India, Greece, and China. He believed something radical for his time: mathematics should be a language, not a collection of tricks.

At a time when every problem seemed to require a different method, al-Khwārizmī imagined a system that could describe *any* unknown quantity in a single, consistent way. His book, al-Jabr wa'l-Muqābala "Restoring and Balancing"—became the world's first true algebra text. From it we inherit the word algebra, the widespread use of the digits 0–9, and even the word algorithm, which comes from his name. For the first time, mathematicians could write ideas like x, x^2, and combinations of known and unknown quantities. Without this symbolic language, nothing that follows in calculus would be possible.

But algebra's story did not end there.

A century later in Persia, the poet-astronomer Omar Khayyam confronted a problem that resisted all known algebraic methods: cubic equations. Quadratics could be solved cleanly, but cubics behaved differently. Rather than forcing them into algebra, Khayyam turned to geometry, solving cubic equations by finding intersections of curves—circles, parabolas, and hyperbolas. He was the first to truly understand that third-degree equations lived in a different mathematical world.

Five hundred years later, during the Renaissance, this challenge ignited a mathematical race in Italy. Tartaglia discovered partial rules for solving cubics. Gerolamo Cardano published the complete algebraic solution in his famous book Ars Magna (1545). But something extraordinary—and disturbing—happened. When applying these formulas, Cardano found expressions involving the square root of –1. Such numbers seemed impossible, meaningless, even absurd.

Yet the answers worked.

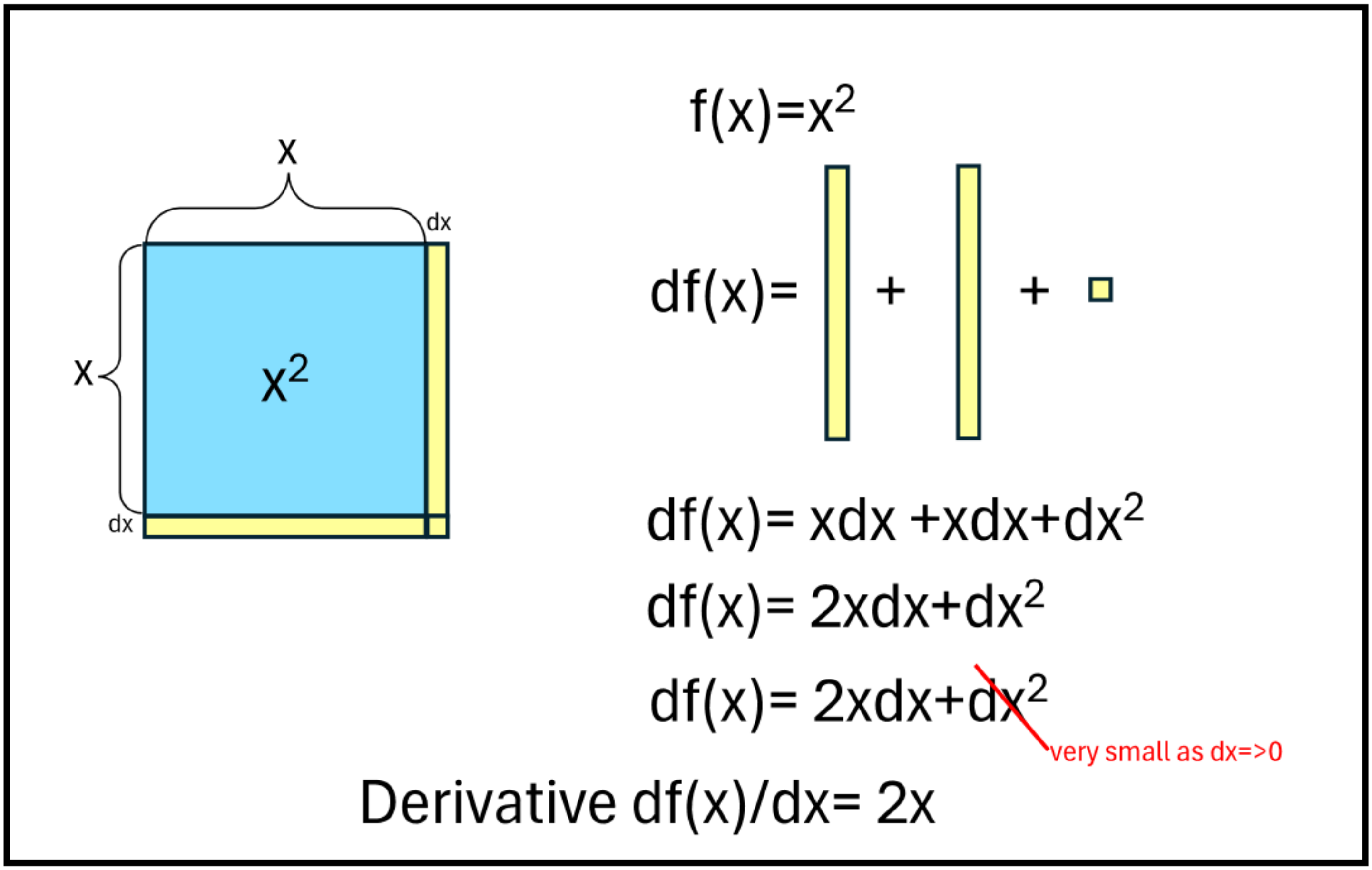
Enter Rafael Bombelli, who made a bold decision: instead of rejecting these strange quantities, he developed clear rules for using them. In doing so, he gave mathematics a new kind of number—what we now call imaginary numbers. They were not invented on purpose; they were discovered accidentally while trying to solve cubic equations. Algebra had grown powerful enough to push beyond common sense.

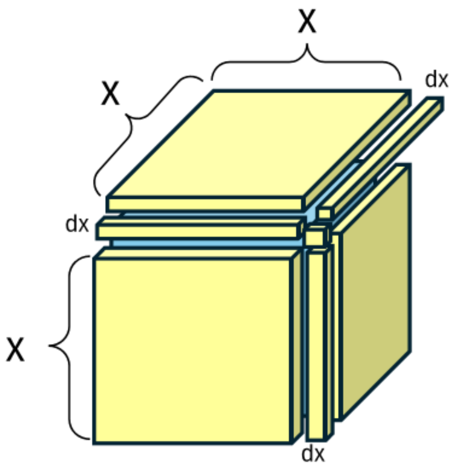
As algebra matured, mathematicians across Europe began asking a new question: *How do quantities change?* Fermat used algebra to approximate slopes. Torricelli sliced curves into infinitely thin pieces to measure areas and volumes. The Bernoulli brothers used algebraic rules—especially the Power Rule—to solve problems about motion, growth, and physical forces.

Finally, in the late seventeenth century, Newton and Leibniz brought all these ideas together into a new subject: calculus. Leibniz introduced the notation students still use today:

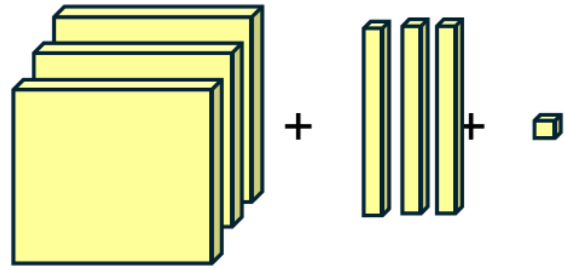
$$\frac{d(x^2)}{dx} = 2x \text{ and } \frac{d(x^3)}{dx} = 3x^2$$

📸 Student Worksheets Solutions





$f(x)=x^3$

$df(x)=$ 

$$df(x)= 3x^2dx +3xdx^2+dx^3$$

$$df(x)= (3x^2+3xdx+dx^2)dx$$

very small as dx=>0

Derivative $df(x)/dx= 3X^2$

📘 Adventure 3 — Student Worksheet Instructions

Discovering the Power Rule by Watching Shapes Grow

In this activity, you will discover where derivatives come from by watching how areas and volumes change when a shape grows just a tiny amount.

You are not memorizing rules.

You are seeing them happen.

Work carefully, label clearly, and explain your thinking in words when asked.

🔹 Part A — Growing a Square: $f(x) = x^2$

Look at the square diagram on (Top half) of your Student Worksheet Diagrams Parts A and B.

The original square has side length x and area x^2.

The square grows slightly: each side increases by a very small amount called dx.

Mark dx on the right side and the bottom side of the square.

Now focus on the *new pieces* that appear:

One thin vertical strip

One thin horizontal strip

One tiny corner square

Your tasks:

Write an expression for the area of each new piece.

Add these pieces together to find df(x), the change in area.

Simplify your expression for df(x).

Divide by dx to find df(x)/dx.

Decide which terms become very small as $dx \to 0$ and explain why.

When finished, write the derivative at the bottom of the page.

🔹 Part B — Growing a Cube: $f(x) = x^3$

Now look at the cube diagrams (bottom half) of your Student Worksheet Diagrams Parts A and B.

The original cube has side length x and volume x^3.

The cube grows slightly in all directions by dx.

Identify all the new pieces that appear.

You should find:

Flat sheets added to faces

Rectangular edge pieces

A tiny corner cube

Your tasks:

Write expressions for the volume of each type of added piece.

Add them together to find df(x).

Factor your expression so dx appears clearly.

Divide by dx to find df(x)/dx.

Decide which terms disappear as dx → 0 and explain why.

Write the final derivative clearly at the bottom.

🔍 What to Pay Attention To

Bigger shapes gain more area or volume, even if dx is the same.

The most important terms are the ones that are proportional to dx.

Tiny terms like dx^2 and dx^3 become negligible for very small dx.

The number in front of x tells you how many directions the shape grows.

⭐ What You Should Discover

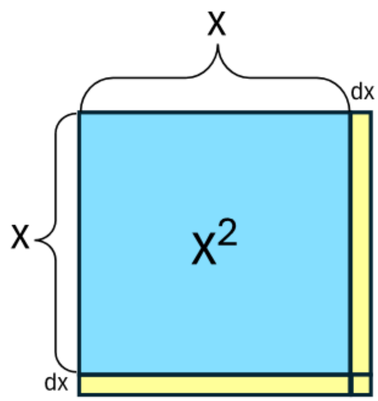
By the end of this activity, you should be able to explain—in words and pictures—why:

The derivative of x^2 is 2x

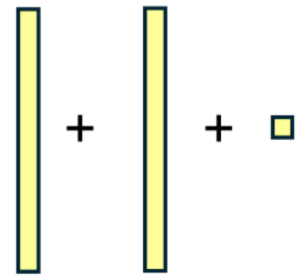
The derivative of x^3 is $3x^2$

This idea is called the Power Rule, and you have discovered it geometrically.

📷 Adventure 3 — Student Worksheet Diagrams Part A and B



$f(x)=x^2$

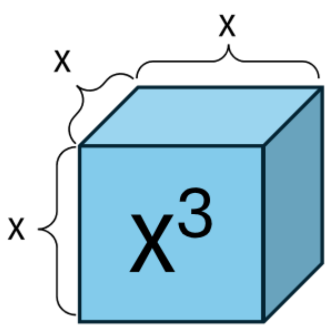
df(x)= 

df(x)=

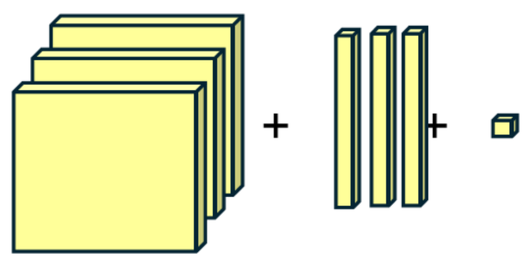
df(x)=

df(x)=

Derivative df(x)/dx=

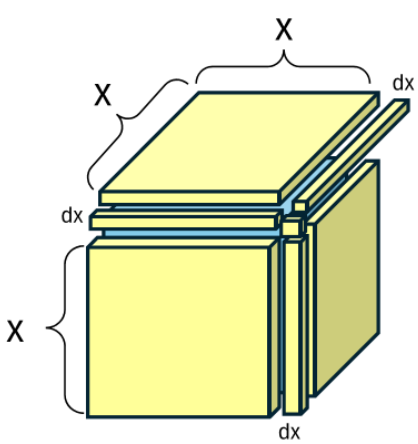


$f(x)=x^3$

df(x)= 

df(x)=

df(x)=



Derivative df(x)/dx=

🧑‍🏫 Adventure 4 – Free Fall – Gravity and the Birth of Calculus

🧩 Purpose

Students explore how Galileo's and Newton's studies of falling objects naturally lead to the key ideas of calculus: velocity as the derivative of position and distance as the integral of velocity. This Adventure connects constant acceleration to the birth of differential and integral calculus.

🧱 Prerequisites

Students should already understand uniform acceleration (Section 2) and be able to interpret distance–velocity–acceleration charts.

🎬 Watch the Video:

[Motion in a Straight Line: Crash Course Physics #1]

This video provides a clear introduction to kinematics: position, velocity, and acceleration. Emphasize the relationships between these quantities, especially how velocity and acceleration describe changes in motion. This video prepares students for DiVA-style reasoning and graph-based analysis.

📊 Set-up

- Student Activity Sheet — Section 3 Free Fall
- Charts — Free Fall from 980 m (14 s)
- Annex story 'The Apple, the Bet, and the Book

🧮 Guided Procedure

1️⃣ Start with the Story —

Read 'The Apple, the Bet, and the Book' to set the historical scene and spark curiosity.

2️⃣ Connect to Real Motion —

Discuss a free fall from 980 m (≈ 14 s). Mark positions at 1-second intervals; height $\propto t^2$.

3️⃣ Chart Exploration —

Use x(t), v(t), a(t) charts to show how slope and area connect. Highlight that velocity is the slope of distance and distance is the area under velocity.

4️⃣ Differentiate and Integrate Visually — $d/dt\ (t^2) = 2t$ → velocity; $\int t\ dt = ½t^2$ → distance.

5️⃣ Discuss Constant Acceleration — $a = g = 9.8\ m/s^2$; $v = g\ t$; $x = ½\ g\ t^2$.

6️⃣ *Ask why the area under v(t) equals distance.*

7️⃣ *Link to Calculus Language — $\Delta x/\Delta t \to v$ and $\Sigma\ v\ \Delta t \to x$.*

8️⃣ Reflect — Motion is the first and most intuitive bridge to calculus thinking.

🧠 Discussion and Reflection

• Why does velocity increase linearly while distance curves quadratically?
• What would change if air resistance were added?
• How does this mirror Galileo's experiments on inclined planes?

🧩 Optional Extensions

Add optional enrichment or challenge activities here.

👩‍🏫 Teacher Tips

Encourage students to trace slopes on the distance curve and shade areas under velocity. Ask them what each graph "tells" about the story of the fall. Relate the moment of impact (≈ 10 s) to 980 m to validate the model.

🪄 Wrap-Up/Key takeaways

Free fall illustrates the core ideas of calculus: the derivative as instantaneous rate of chan ge and the integral as accumulated effect. This lesson bridges Galileo's curiosity with New ton's laws — the foundation of Calculus Mechanics.

🔧 Extension Points for Teachers

• Compare Galileo's inclined plane with vertical fall.

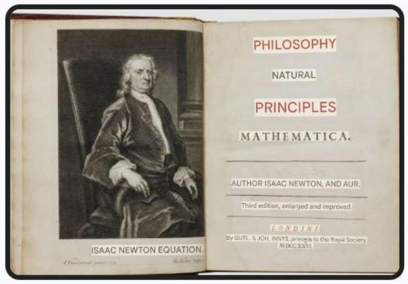
• Graph v^2 vs x to see the linear relation $2\ a\ x = v^2$.

• Preview energy concepts ($½\ m\ v^2$) for Section 4.

• Connect to the Power Rule discussion from Math Circle 12 notes.

🌳 Adventure 4 Story – The Apple, the Bet, and the Book

The Bet: Edmund Halley, Robert Hooke, and Christopher Wren, _Trajectory of Planets Around the Sun



Principia Mathematica

Sir Isaac Newton

Long ago in England, there lived a quiet young man named **Isaac Newton** who loved thinking more than talking. When the plague closed London's universities, he went home to the countryside. There, sitting beneath an apple tree, he began wondering *why* an apple always fell straight down—never sideways or up.

He realized the same invisible pull that brought the apple to the ground must also hold the **Moon** in its path around Earth.

Years later three brilliant friends—**Sir Christopher Wren**, **Robert Hooke**, and **Edmond Halley**—were arguing about what kind of curve a planet follows around the Sun. Wren, always mischievous, offered a **bet**: "A golden cup to whoever can prove the path of the planets!"

Hooke bragged that he knew the answer but did not want to spoil the bet.

Halley, curious and determined, rode all the way to Cambridge to ask Newton. When he said, "What shape would the orbit be if gravity pulls as the inverse square of distance?" Newton quietly replied,

"An ellipse, of course—I worked it out years ago." Halley was stunned. He begged Newton to find his papers. When Newton couldn't, he simply rewrote everything—from scratch—adding new ideas that tied the heavens and the Earth together.

The result was a great book with a Latin title: ***Philosophiæ Naturalis Principia Mathematica***, or simply **The Principia**.

It showed that the same simple rules describe falling apples and whirling planets. In developing these ideas, Newton also invented calculus — the powerful mathematical language needed to describe motion and change. He formulated his three laws of motion and the universal law of gravitation, revealing that the same force pulling an apple to Earth also governs the motion of the Moon and planets. Halley even paid to publish it, and Newton's work transformed our view of the universe—from a place of mystery and speculation into one governed by precise, testable laws—making physics truly scientific for the first time.

✨ Moral ✨

Sometimes one person's curiosity, one friend's encouragement, and one friendly bet can move the heavens themselves.

📘 Adventure 4 – Student Worksheets Solutions

Free Fall — 980 m Drop ($g = 10\ m/s^2$)

No friction · Constant acceleration downward · Time ticks every 1 second
(from rest, thrown out of a plane — no parachute!)

✳ Equations of Motion

$y(t) = 980 - ½ g t^2$, $v(t) = y'(t)$, $a(t) = v'(t)$

1. Height vs Time

Height after 4 seconds: $y(4)=980-5\cdot(4^2)=980-80=900$ m

Height after 10 seconds: $y(10)=980-5\cdot(10^2)=980-500=480$ m

Farther fallen between 4 s and 10 s: 900−480 = 420 m

2. Velocity and Distance

Velocity function (downward speed): $v(t)=g\,t = 10t$ (m/s).

Area under v(t) from 0 to 4 s (triangle): ½·4·40 = 80 m.

Area under v(t) from 0 to 10 s (triangle): ½·10·100 = 500 m.

Does this match the distance fallen on the height curve? YES.
Because $980-y(4)=980-900=80$ and $980-y(10)=980-480=500$.

3. Hitting the Ground

Solve $y(t)=0$: $980-5t^2=0 \Rightarrow 5t^2=980 \Rightarrow t^2=196 \Rightarrow t=14$ s (take positive time).

Time that flyer hits the ground: 14 s. Check on graph: correct (near 14 s).

Velocity at impact: $v(14)=10\cdot14 = 140$ m/s.

Convert to km/h: 140 m/s × (3600 s / 1 h) × (1 km / 1000 m) = 140×3.6 = 504 km/h.

4. Acceleration Check

Acceleration is constant: $a(t)=g=10\ m/s^2$ (downward).

Rectangle under a(t) from 4 to 10 seconds: width = 6 s, height = $10\ m/s^2$.

Area = 10×6 = 60 (m/s). Explanation: Area under acceleration gives change in velocity (Δv). Here Δv = 60 m/s, and indeed $v(10)-v(4)=100-40=60$ m/s.

*** Calculus (Show your work — remember Derivatives and Integrals)

Given: $y(t)=980-½\cdot g\cdot t^2$ with $g=10$, so $y(t)=980-5t^2$.

Differentiate to get velocity: $v(t)=y'(t)=-10t$ (m/s).

Interpretation: the negative sign means y is decreasing (moving downward). The speed is $|v(t)|=10t$.

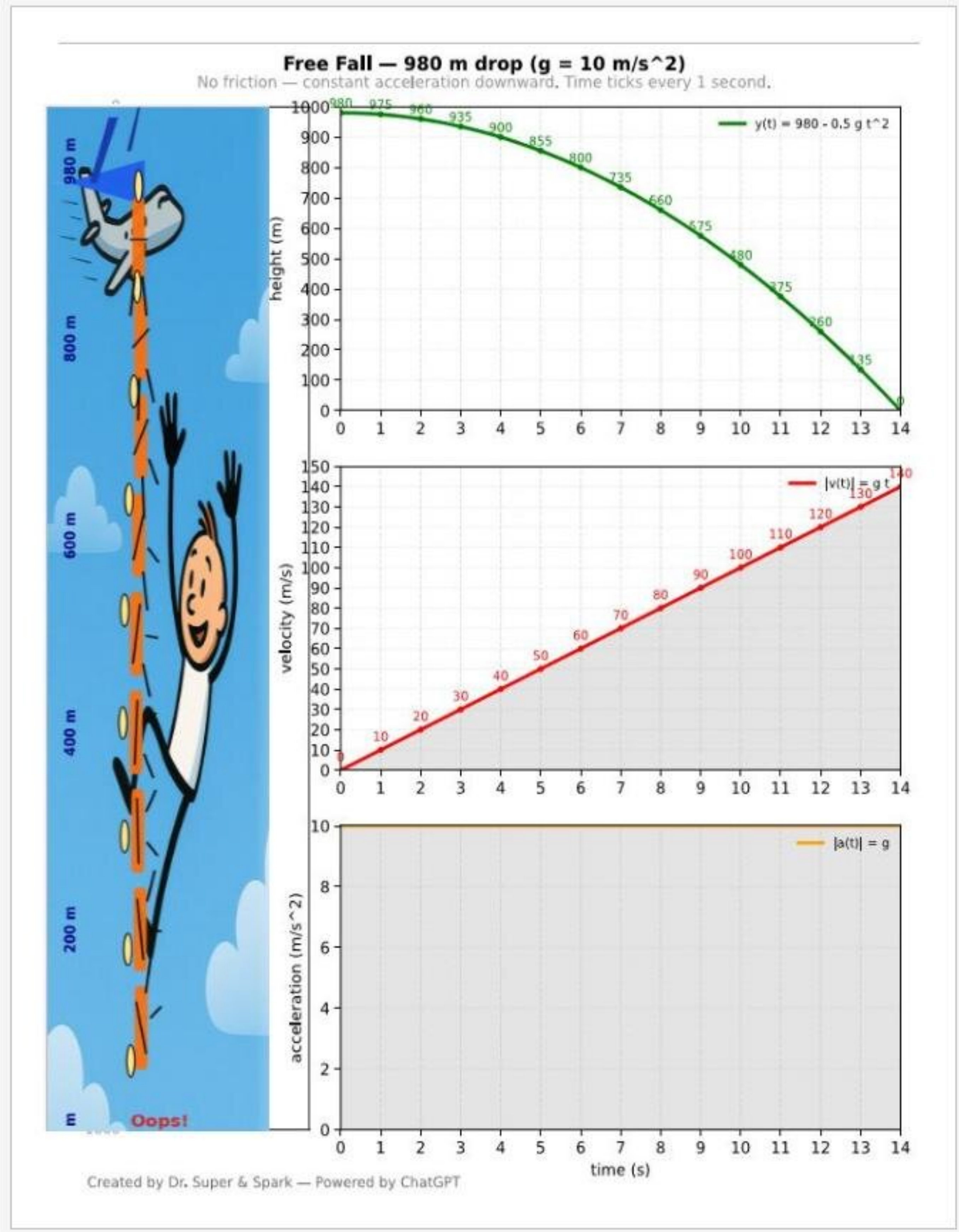
Differentiate again to get acceleration: $a(t)=v'(t)=-10\ (m/s^2)$. Magnitude is g=10 downward.

Could you have found y(t) and v(t) from v(t) and a(t)? Yes—by integrating and using initial conditions.

1) From acceleration to velocity: $v(t)=\int a(t)dt = \int(-10)dt = -10t + C$. Since $v(0)=0$, $C=0$, so $v(t)=-10t$.

2) From velocity to position: $y(t)=\int v(t)dt = \int(-10t)dt = -5t^2 + C$. Since $y(0)=980$, $C=980$, so $y(t)=980-5t^2$.

📈 Adventure 4 – Student Chart: Distance-Velocity-Acceleration (DiVA) Charts



📘 Adventure 4 – Student Activity Sheets

Free Fall — 980 m Drop (g = 10 m/s^2)

No friction · Constant acceleration downward · Time ticks every 1 second

(from rest, thrown out of a plane — no parachute!)

✳ Equations of Motion

$$\boldsymbol{y(t) = 980 - \frac{1}{2}gt^2, v(t) = gt, a(t) = g}$$

🧩 Questions for Young Scientists

1. **Height vs Time**
   - Use the graph to find the height after 4 seconds and 10 seconds.
     - Height After 4 seconds = _____________
     - Height After 10 seconds =_____________
   - How much farther did our flyer fall between 4 s and 10 s? _____________
2. **Velocity and Distance**
   - Estimate the *area under the velocity curve* from 0 to 4 s and 0 to 10 s.
     - Area under the Velocity Curve = ________________
   - Does this area match the *distance fallen* on the height curve?
     - Yes No
3. **Hitting the Ground**
   - Set $y(t) = 0$. At what time t does the flyer hit the ground?
     - Set y(t) = 0 and solve $\boldsymbol{y(t) = 980 - \frac{1}{2}gt^2}$
     - Time that Flyer Hits the ground: ____________
   - Check the Graph to see if your answer is correct.
     - Check _________
   - What is the velocity at that instant in m/s and in km/h? velocity when he hits the ground = _____________(m/s)
   - Use 1Km/1000m and 1h/3600s unitary factors to find the velocity in Km/h

- velocity (Km/h) = ______________________________________ (Km/h)

4. **Acceleration Check**

   - Draw a rectangle under the constant-acceleration line between 4 and 10 seconds.

     Area =______________

   - What is the area of this rectangle. How does it relates to Velocity.
   - EXPLAIN: ____

1. ***** Calculus (Show your work – remember Derivates and Integrals)**

$$\boldsymbol{y(t) = 980 - \frac{1}{2}gt^2}$$

$$\boldsymbol{, v(t) = gt,}$$

$$\boldsymbol{a(t) = g}$$

Drive the equations for v(t) and a(t) from y(t) and a(t) respectively

Could you have found y(t) and v(t) from v(t) and a(t) respectively? How?

🦸 Adventure 5 – From Force to Kinetic Energy

🧩 Purpose

Students review the equations of motion for Drag Racing and discover the similarity of these equations with the Free Fall equations in Adventure 4. They also connect Newton's Second Law, the concept of work, and kinetic energy. They review distance–velocity–acceleration (DiVA) charts, recognize that area under v(t) gives distance, and slope of x(t) gives velocity, then derive KE = ½ m v^2.

🧱 Prerequisites

Familiarity with constant acceleration motion (drag racing ¼-mile, a = 8 m/s^2). Ability to read values from x–t, v–t, a–t charts.

🎬 Watch the Video: Work, Energy, and Power – Crash Course Physics #9

In this video, you'll see how motion and energy are connected, and how an object's energy increases with speed in a way that is not simply linear, introducing the idea of kinetic energy. As you watch, think about how energy changes when velocity doubles, how constant acceleration affects energy over time, and how distance, velocity, and energy might all be connected through math.

📊 Set-Up / Materials

Three-chart DiVA page (distance, velocity, acceleration vs time), Projector or printouts for charts.

Student activity sheet:

Part 1: 'Equations of Motion for Drag Racing',

Part 2: 'Kinetic Energy and Work' page.

🧮 Guided Teaching Procedure

1 Start with the Story — Read '**Why People Started Drag Racing on a Track (and Not on the Street)**'

You can also read the alternative story: '**Ibn Sīnā (Avicenna): The Persian Doctor Who Discovered Why Things Keep Moving.**'

2 Connect to Real Motion — Discuss drag racing over 400 yards that takes 10 seconds.

3 Chart Exploration —

Use x(t), v(t), a(t) charts to show how slope and area connect. Highlight that velocity is the slope of distan ce and distance is the area under velocity

4 Differentiate and Integrate Visually — d/dt (t^2) = 2t → velocity; ∫ t dt = ½t^2 → distance.

5 Discuss Constant Acceleration — a = 10 m/s^2; v = g t; x = ½ a t^2.

6 *Ask why the area under v(t) equals distance.*

Kinetic Energy and Work

1 Review Work and Kinetic Energy—

This part needs a lot of explanation.

2 Link Force and Work —

Reintroduce Newton's Second Law (F = m a). Define Work as W = F . x, Substitute F = m a to get W = m.a.x . From $v^2 = 2$ a.x, derive W = ½ m v^2.

3 Physical Meaning —

Discuss that work is energy transfer. Both work and kinetic energy share units: Joule (J = N·m Newton-meter).

4 Kinetic Energy and Work Table —

Students fill in values for t = 2, 5, 10 s using a = 8 m/s^2. Guide them to see that W and KE increase proportionally with t^2.

5 Plot Kinetic Energy vs Time —

Students plot the green curve (KE = 32 t^2 kJ). Highlight points at t = 2, 5, 10 s and note that the graph is parabolic. Ask why doubling time quadruples energy.

🧠 Discussion Prompts

• Why does area under the velocity curve represent distance?

• What does the slope of the distance curve represent?

• How do force and distance combine to produce energy transfer?

🧑‍🏫 Teacher Tips

Encourage students to verbalize how the three charts connect. Pause after each graph interpretation. Let students compute and verify their table values before showing the filled version.

🪄 Wrap-Up — Key Takeaway

Work done by a net force equals the change in kinetic energy (W = ΔKE). This activity links constant-acceleration motion to energy concepts, preparing students for future calculus integration activities.

🔧 Extension — Real-World Example

A 1000 kg car accelerating to 30 m/s has KE = ½ m v^2 = 450 kJ. Ask how long it would take to reach that energy level if the engine outputs 90 kW (≈ 5 s).

Adventure 5 Story – Why People Started Drag Racing on a Track (Not on Streets)

Long before Teslas or Formula 1, American teenagers rebuilt old Fords and Chevys in their driveways and wanted one thing: **to find out whose car was fastest**. But they raced on normal streets, no rules, no safety crews, no seat belts. Just young drivers, powerful engines, and luck.

The Wake-Up Moment: The most famous young actor of the time was **James Dean**, a real race-driver. Kids copied everything he did. But on his way to an official race, another car turned suddenly, and there was a fatal crash. America realized something important:

Speed isn't the problem. Unsafe speed is.

The Birth of the Drag Strip: Communities came up with a smart idea:

Move racing to controlled tracks: Old airfields and long straight roads became drag strips with: Measured, straight lanes; Timing systems; Safety teams; Rules; Space for spectators.

Street chaos became a real sport—and a science.

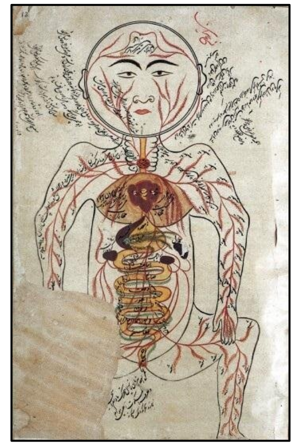
Why Drag Racing Matters in Math Circle: Once racing became safe, drivers and engineers started asking scientific questions: How fast can a car accelerate? Does less mass mean more speed? What does the velocity graph look like? How far do you travel with constant acceleration?

Those are the same charts you're learning to read—distance, velocity, and acceleration. Drag racing makes them real.

A Modern Reminder: Today the biggest danger isn't speed, it's distraction. Two seconds of texting can be more dangerous than driving fast. That's why racing belongs on the track and why understanding physics keeps people safe.

Speed is exciting. **Focus is everything. But u**nderstanding the physics — that's the superpower. Now let's see what your charts say when the light turns green.

Adventure 5 Alternative Story – Ibn Sīnā (Avicenna): The Persian Doctor Who Discovered Why Things Keep Moving

A thousand years ago, in the cities of Bukhara and Isfahan, lived a scientist so brilliant that people called him *The Master of Masters*. His name was **Ibn Sīnā (Avicenna)**. Most people remember him as the greatest physician of the medieval world — but he also made a discovery in **physics** that leads directly to **Newton's First Law**.

Avicenna the Doctor: His medical encyclopedia, *The Canon of Medicine*, became the world's most influential medical text for nearly **600 years**. He described infectious diseases, recommended quarantine, explained diabetes and meningitis, analyzed how emotions affect health, and created remarkably early methods of clinical testing. He saved lives long before modern hospitals existed.

Avicenna the Physicist: Avicenna also asked a question no one had answered correctly:

Why does a moving object keep going after the push ends?

Aristotle believed motion required a constant push. Avicenna rejected this. He argued that when you throw an object, you give it an invisible internal tendency that keeps it moving. He called this force "**mayl**"**(میل)** — a Persian word meaning *inclination* or *desire*.

He explained that: A stronger push creates more "**mayl**"; Heavier and faster objects have more "**mayl**"; Motion stops only because of resistance; The push stays *inside* the object.

This is the earliest clear expression of **momentum** and **inertia** — centuries before Newton.

Why This Matters Today

Avicenna didn't know about race cars or rockets. But he understood the secret behind all motion:

Speed begins with a burst of force — and that force remains in the object until something stops it. It's the same idea engineers use today to design everything from engines to spacecraft.

Transition to Drag Racing Story

Avicenna discovered *why* things keep moving.

The story of Drag racing shows **what humans did when they decided to test that idea at full speed — safely — on a drag strip.**

📘 Student Worksheets Solutions

Drag Racing — 1/4 mile (400 m) No friction · Constant acceleration from rest - Time ticks every 1 second

✳ Equations of Motion

- $x(t) = \frac{1}{2}at^2$
- $v(t) = at$
- $a(t) = a(\text{constant})$

🧩 Questions

1. How far, how fast?

Read the distance traveled at 5 seconds and 10 seconds from the distance vs time graph. What are the speeds at those times?

Distance traveled after 5 seconds: ____________ **Answer: 100 m ($x = ½·8·5^2 = 100$)**

Distance traveled after 10 seconds: ____________ **Answer: 400 m ($x = ½·8·10^2 = 400$)**

2. Area means distance

Look at the triangular area under the velocity curve from 0–5 seconds and 0–10 seconds.
Does each of these areas equal the distance on the distance vs time graph in question 1?

Area under the velocity curve at 5 seconds: ____________ **Answer: 100 m (distance equals area under v–t)**

Area under the velocity curve at 10 seconds: ____________ **Answer: 400 m (distance equals area under v–t)**

3. Slope means velocity

Draw a tangent to the distance traveled curve at t=5 and estimate the slope. Is this number close to the velocity after 5 seconds?

Slope of the tangent line to distance function at t=5: ____________ **Answer: ≈ 40 m/s (matches v(5) = a·t = 8·5)**

Velocity after 5 seconds from Velocity vs Time chart: ____________ **Answer: 40 m/s**

Are these two numbers close or identical   Yes   No **Answer: Yes**

4 Finish line

Solve $\frac{1}{2}\boldsymbol{at^2} = \mathbf{400}$ for t. How long to reach 1/4 mile? (notice a is constant and you can read it from the acceleration chart) **Answer: t = 10 s (solve ½·8·t^2 = 400 → 4t^2 = 400 → t = 10 s)**

What is $\boldsymbol{v = at}$ at the finish (m/s and km/h)? Velocity at finish: m/s

Answer: 80 m/s (v = a·t = 8·10); 288 km/h (×3.6)

(Use 1Km/1000m and 1h/3600s unitary factors to find the velocity in Km/h)

Velocity at finish = =__ ____km/h

Answer: 80 m/s (v = a·t = 8·10); 288 km/h (×3.6)

5 Acceleration check

The acceleration graph is a flat line. What does the rectangle's area from 0–t tell you about the change in velocity?

What is the change in velocity from 0 to 5 seconds: ____________ **Answer: 40 m/s (Δv = a·t = 8·5)**

What is the change in velocity from 0 to 10 seconds: ____________ **Answer: 80 m/s (Δv = 8·10)**

6 (***) Calculus

Take derivatives to show that you can find the velocity and acceleration functions from distance and velocity functions.

- $x(t) = \frac{1}{2}at^2$ d(x)/dt = **Answer: v(t) = a·t**
- $v(t) = at$ d(v)/dt = **Answer: a (constant)**
- $a(t) = a$(constant)

Take the antiderivative (integral) of acceleration and velocity functions to get the velocity and distance functions.

- $x(t) = \frac{1}{2}at^2$
- $v(t) = at$ ∫v(t)dt = ∫atdt = a∫tdt = **Answer: ½·a·t^2 = x(t)**
- $a(t) = a$ ∫a(t)dt = ∫adt = a∫1dt = **Answer: a·t = v(t)**

From Force to Kinetic Energy — How Motion Gets Its Energy

Work, Newton's Second Law, motion equations, and the energy connection

1) Work and the Joule

Work (W) happens when a force moves something through a distance: $W = F \times d$.

Unit of Work: Joule (J). 1 J = work by 1 N over 1 m.

Example: lifting a small apple (~100 g) upward by 1 m takes about 1 J.

2) Force and Newton's Second Law

A force is a push or pull that changes motion. Newton's 2nd Law says: $F = m a$.

Unit of Force: Newton (N). 1 N makes 1 kg accelerate by 1 m/s^2.

Example: lifting a 1 kg book straight up requires ~10 N of force.

3) Equations of Motion (starting from rest, constant a)

For motion starting from rest with constant acceleration:

$v = a t$

$s = \frac{1}{2} a t^2$

$v^2 = 2 a s$

4) From Work to Kinetic Energy

Multiply $v^2 = 2 a s$ by $m/2$ → $(1/2)\, m v^2 = m a s = F s = W$.

So the work done on an object becomes its kinetic energy: $KE = (1/2)\, m v^2$.

5) Examples — Drag Racer and Falling Person (no air resistance)

Drag racer ($m = 1000$ kg, $a = 8\ m/s^2$): after 10 s, $v = 80$ m/s, $s = 400$ m, KE = 3.2 MJ.

Falling person ($m = 70$ kg) from $h = 980$ m, with $g = 10\ m/s^2$:

$v^2 = 2 g h = 2 \times 10 \times 980 = 19{,}600 \rightarrow v \approx 140$ m/s; KE ≈ 0.69 MJ.

This is roughly one-fifth of the drag racer's kinetic energy at 10 s.

6) Summary

Forces do work; that work becomes kinetic energy of motion: $KE = \frac{1}{2} m v^2$.

📘 Adventure 5 – Kinetic Energy and Work – Student Exercise

- Work is force times distance **W = Fd** (units Joules)
- Force relates to motion **F = ma** (Newtons second Law) (units Newton)
- Substituting **F** in the Work formula we get **W=mad →W/m = ad**
- Using the equations of motion **v = at, d = ½ a. t²** we find that The **Kenetic Energy KE** that is defined by the work that is necessary to get an object of mass m to speed v is **KE = ½ mv²**
- **$v = at \rightarrow v^2 = a^2t^2$** now use **$d = \frac{1}{2} at^2$** to get **$v^2=2ad \rightarrow \frac{1}{2} mv^2= W$**
- Then complete the table below to verify That **KE = Work** .

• **Time (s)**	• **Velocity (m/s)**	• **Distance (m)**	• **KE (kJ)**	• **Work (kJ)**
• 2	• 16	• 16	• 128	• 128
• 5	• 40	• 100	• 800	• 800
• 10	• 80	• 400	• 3200	• 3200

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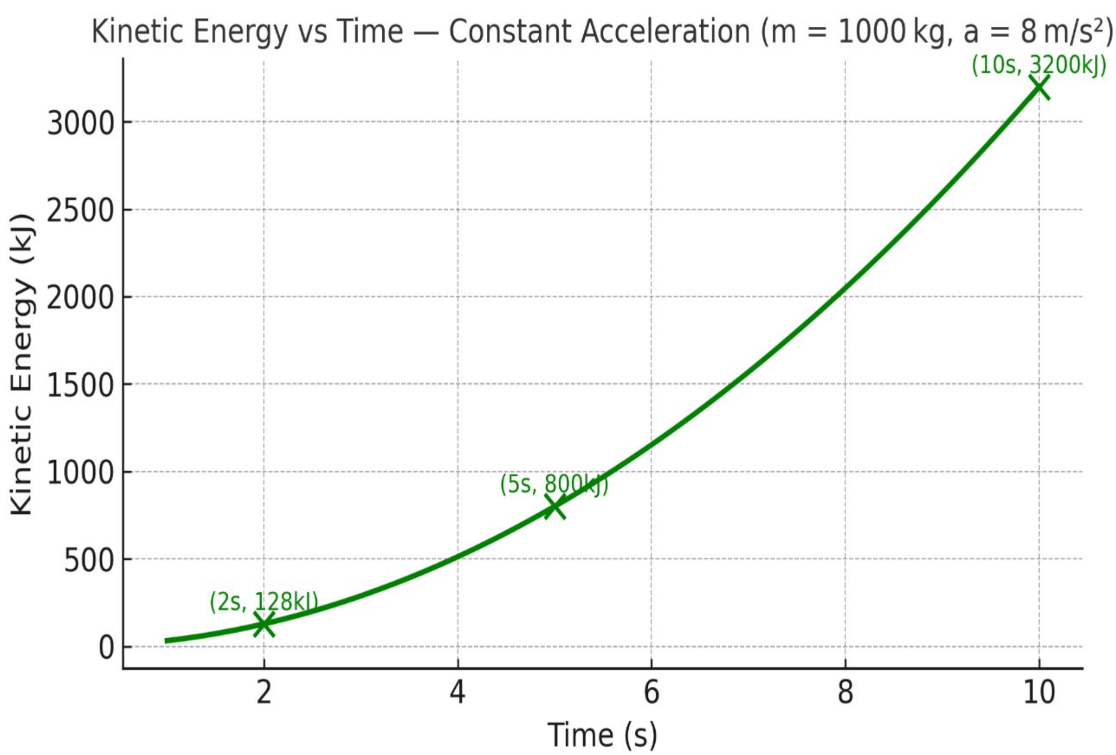

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- Created by Dr. Super & Spark — Powered by ChatGPT

📘 Adventure 5 – Teacher Notes & Discussion

Overview

Students reviewed how distance, velocity, and acceleration relate under constant acceleration. They practiced reading distance from area under v–t, and recognizing velocity as the slope of x–t.

From Force to Kinetic Energy

Start with F = m·a and W = F·Δx. For constant acceleration, W = m·a·Δx. From $v^2 = 2 \cdot a \cdot \Delta x$, we get $a \cdot \Delta x = v^2/2$, hence $W = ½ \cdot m \cdot v^2$. Interpretation: work done by a constant net force equals the change in kinetic energy.

Units

Force: newton ($N = kg \cdot m/s^2$). Work/energy: joule ($J = N \cdot m = kg \cdot m^2/s^2$).

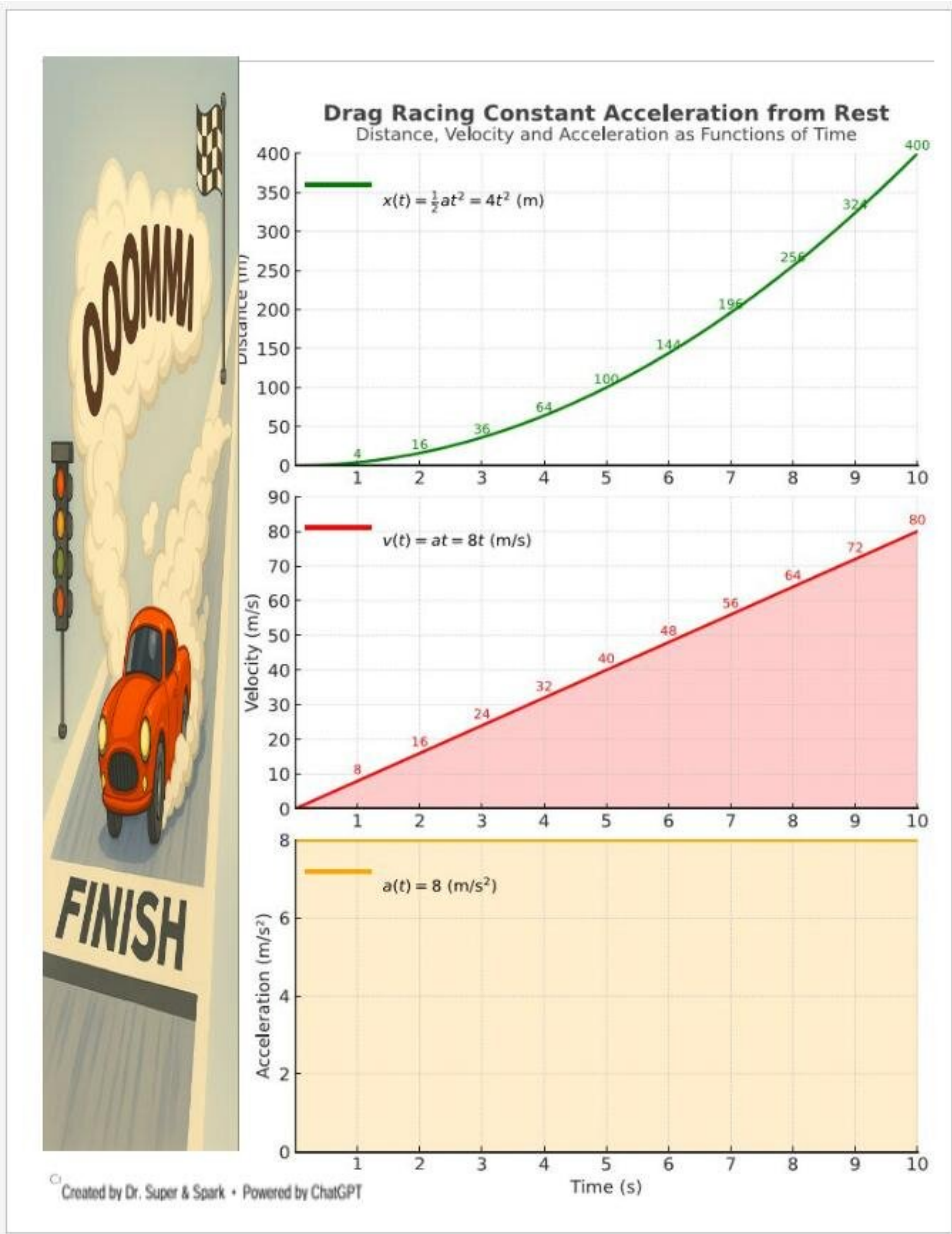
Example

1000 kg car at 30 m/s: $KE = ½ \cdot m \cdot v^2$ = 450,000 J (450 kJ).

Prompts

Ask: Where is distance under the v–t curve? Where is Δv under the a–t curve? Have each student restate the KE derivation in their own words.

Adventure 5 – Student Chart: Distance-Velocity-Acceleration (DiVA) Chart



📘 Adventure 5 – Student Worksheets

Drag Racing — 1/4 mile (400 m)-No friction-Constant acceleration from rest-Time ticks every 1 second

✳ Equations of Motion

- $x(t) = \frac{1}{2}at^2$
- $v(t) = at$
- $a(t) = a$(constant)

🧩 Questions

4. **How far, how fast?**

Read the distance traveled at 5 seconds and 10 seconds from the distance vs time graph. What are the speeds at those times?

Distance traveled after 5 seconds: ____________

Distance traveled after 10 seconds: ____________

5. **Area means distance**

Look at the triangular area under the velocity curve from 0–5 seconds and 0–10 seconds.
Does each of these areas equal the distance on the distance vs time graph in question 1?

Area under the velocity curve at 5 seconds: ___________

Area under the velocity curve at 10 seconds: ____________

6. **Slope means velocity**

Draw a tangent to the distance traveled curve at t=5 and estimate the slope. Is this number close to the velocity after 5 seconds?

Slope of the tangent line to distance function at t=5: ___________

Velocity after 5 seconds from Velocity vs Time chart: __________

Are these two numbers close or identical **Yes** **No**

No friction · Constant acceleration from rest · Time ticks every 1 second

4 Finish line

Solve $\frac{1}{2}\boldsymbol{at^2} = \mathbf{400}$ for t. How long to reach 1/4 mile? (notice a is constant and you can read it from the acceleration chart)

What is $\boldsymbol{v} = \boldsymbol{at}$ at the finish (m/s and km/h)? Velocity at finish: m/s

(Use 1Km/1000m and 1h/3600s unitary factors to find the velocity in Km/h)

Velocity at finish = =__ ____km/h

5 Acceleration check

The acceleration graph is a flat line. What does the rectangle's area from 0–t tell you about the change in velocity?
What is the change in velocity from 0 to 5 seconds: ____________
What is the change in velocity from 0 to 10 seconds: ____________

6 (***) Calculus

Take derivatives to show that you can find the velocity and acceleration functions from distance and velocity functions.

- $x(t) = \frac{1}{2}at^2$ d(x)/dt = _____________
- $v(t) = at$ d(v)/dt = ______________
- $a(t) = a$(constant)

Take the antiderivative (integral) of acceleration and velocity functions to get the velocity and distance functions.

- $x(t) = \frac{1}{2}at^2$
- $v(t) = at$ ∫v(t)dt = ∫atdt = a∫tdt = _______________
- $a(t) = a$ ∫a(t)dt = ∫adt = a∫1dt = _______________

7. (***) Kinetic Energy and Work

Work is force times distance **W = Fd** (units Joules)

Force relates to motion **F = ma** (Newtons second Law) (units Newton)

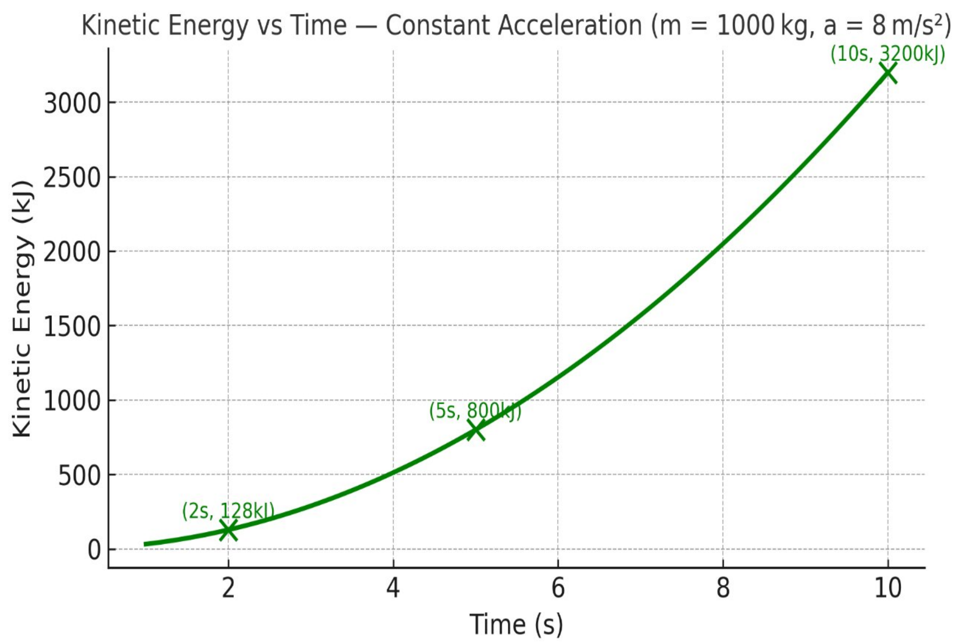
Substituting **F** in the Work formula we get **W=mad ➔W/m = ad**

Using the equations of motion **v = at, d = ½ a. t²** we find that The **Kenetic Energy KE** that is defined by the work that is necessary to get an object of mass m to speed v is **KE = ½ mv²**

$v = at \rightarrow v^2 = a^2t^2$ now use $d = \frac{1}{2} at^2$ to get $v^2 = 2ad \rightarrow \frac{1}{2} mv^2 = W$

Then complete the table below to verify That **KE = Work** .

Time (s)	Velocity (m/s)	Distance (m)	KE = ½ m v² (kJ)	Work = F × d (kJ)
2				
5				
10				



🧑‍🏫 Adventure 6 – Area Under the Velocity Curve & Meaning of the Integral

🧩 Purpose

Students explore how **distance traveled** can be determined by the **area under the velocity curve**, and how **velocity** itself is the **area under the acceleration curve**. This activity leads directly into the **Fundamental Theorem of Calculus**, connecting **derivatives** and **integrals** through a real physical interpretation.

🧱 Prerequisites

Students should already have:

- Basic understanding of **distance, velocity, acceleration** relationships from earlier DiVA activities.
- Experience reading values from line graphs.
- Informal exposure to **derivatives as slopes**.
- They **do not yet need** formal integration; this activity *develops* the need for it.

📖 Read the Story Galileo and the Secret of Motion

How adding small pieces revealed the laws of nature

Galileo asked a simple but powerful question: *How does motion build up over time?* By slowing motion down, he discovered that distance comes from adding many small pieces—just like area under a curve.

🎥 Watch the Video

- 3Blue1Brown — Integration and the fundamental theorem of calculus | Chapter 8, Essence of calculus

  This video reveals one of the deepest ideas in calculus: finding areas and taking derivatives are inverse processes. As you watch, notice how adding up small pieces (area) can be undone by taking a rate of change (derivative), and vice versa. This is the Fundamental Theorem of Calculus — showing that accumulation and change are not just related, but are two operations that reverse each other.

  Students in Math Circle responded extremely well to pausing & discussing key steps.

📊 Set-Up / Materials

- Student worksheet pages for **rectangular area estimates** and **calculus confirmation**.
- The **three-chart page** (DiVA Chart) showing distance, velocity, and acceleration vs. time.
- Highlighters or colored pencils recommended.
- Optional: extra scratch paper for summation steps.

🧮 Guided Teaching Procedure for Activities 1 and 2

✏️ Activity 1 — Area under the Velocity Curve (Teacher Guide)

This activity introduces students to one of the most important ideas in calculus:

👉 **area under a velocity curve represents total distance traveled.**

Students begin with a familiar approach — reading values directly from the graph — and then transition to estimating area using rectangles. This builds an intuitive bridge from arithmetic to integration.

🎯 Learning Goals

Connect **velocity → distance** through area

Understand **rectangle approximations** as estimates of area

See how smaller intervals improve accuracy (**convergence**)

Compare estimates with the **exact integral result**

🧠 Key Ideas to Emphasize

The velocity function is:

$$v(t) = t(8 - t)$$

which starts at 0, rises to a peak, then returns to 0.

The **total distance traveled from 0 to 8 seconds** is:

$$\text{distance} = \int_0^8 v(t)\,dt = 85\frac{1}{3} \approx 85.33$$

Rectangle estimates give:

Δt = 1 → **84 m**

Δt = 0.5 → **≈ 85.25 m**

Δt = 0.25 → **≈ 85.31 m**

👉 These values clearly show:

As Δt gets smaller, the estimate gets closer to the true value.

📊 Teaching Flow

Start with the DiVA Charts

Ask: *"How far did the car travel?"*

Students will likely try to read values — this is intentional.

Introduce area

Point to shaded regions under the velocity curve.

Emphasize:
👉 *"This area represents total distance."*

Rectangle approximation

Use Δt = 1 table first

Students compute:

$$\text{distance} \approx \sum v(t) \cdot \Delta t = 84$$

Refinement

Show finer partitions (Δt = 1/2, 1/4)

Highlight improvement without changing the idea

Exact result (calculus reveal)

Show the integral at the bottom

Reinforce:

Rectangles → limit → exact area

⚠️ Common Student Issues

Thinking velocity itself is distance

Forgetting to multiply by Δt

Not recognizing why smaller rectangles improve accuracy

Confusing graph height with area

💡 Key Insight to Highlight

"Distance is not just how fast you go — it is how long you go at that speed."

This naturally leads to:

$$\text{distance} = \int v(t)\, dt$$

✏️ Activity 2— Car Starting, then going fast, Then Stopping

1. Connect the chart to the Story

Ask: "What is happening to the car during this motion? When is it speeding up? When is it slowing down?"

Students should notice:

- **Velocity is rising**, so the car speeds up.
- **Acceleration crosses zero and becomes negative → the car is braking**.

This observation *should come from the children*, not be told.

2. Connect the Three Charts

Use the chart side-by-side:

- **Acceleration → Velocity** is **area** under the acceleration curve.
- **Velocity → Distance** is **area** under the velocity curve.

Tell students:

"Whenever we move UP one chart, we use slope (derivative).
Whenever we move DOWN one chart, we use AREA (integral)."

This is the heart of the **Fundamental Theorem of Calculus** in visual form.

3. Rectangular Approximation — Let Them Discover Convergence

Students complete the table using **dt = 1 seconds**, then **1/2**, then **1/4**.
As interval width shrinks, **their estimated total distance converges** to the true value.

You can guide them by asking:

"Are your estimates getting closer to one specific number?"

In Math Circle:

- Students reached **84 m** with the coarse estimate.
- The exact integral result is **85.33 m**.
  They saw their estimates **approach** the exact result.

Calculus_Mecanics_Activity5

This connection is the *entire conceptual win* of the lesson.

4. Calculus Page (Last Page)

Students compute:

- $x(t) = 4t^2 - \frac{1}{3}t^3$
- $v(t) = 8t - t^2$
- $a(t) = 8 - 2t$

And check:

- Derivative of distance gives velocity.
- Derivative of velocity gives acceleration.

Tell them: “The answers are already in the graphs — calculus is just the language that confirms what you already understand.” This reduces fear & reinforces intuition.

🚀 Extension (optional discussion)

Why is the graph symmetric?

Where is velocity maximized?

How does acceleration relate to the slope of v(t)?

Preview:

👉 “Area under acceleration gives change in velocity”

🧩 Big Picture Connection

This activity sets up the **Fundamental Theorem of Calculus**:

Derivative → slope (velocity from position)

Integral → area (distance from velocity)

Students are seeing both ideas **in motion**.

🧠 Discussion Prompts

Ask at the end:

1. *Why does area under the curve represent something physical?*
2. *Why did we get a better distance estimate when dt got smaller?*
3. *Why is braking shown by **negative acceleration**, not negative velocity?*

🧩 Extension (Optional)

- Introduce **signed area** under acceleration → velocity change.
- Let students compare *average velocity* to *instantaneous velocity* at midpoints.
- Preview the **FTC** verbally:

“Differentiation and integration undo each other.”

🧑‍🏫 Teacher Tips

- Encourage **verbal explanation** from students before using formal notation.
- Let them struggle productively with the rectangles — **do not jump to the integral too quickly**.
- When a student notices convergence, **pause and celebrate** — that is the conceptual milestone.
- If a student gets lost ,work *one rectangle at a time* and narrate slowly.

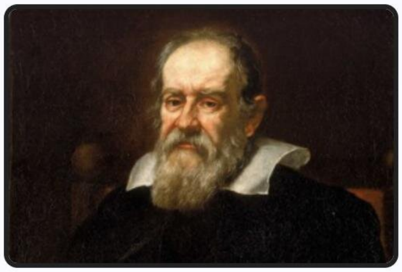
🪄 Wrap-Up — Key Takeaway

Distance, velocity, and acceleration are three levels of the same story.
Derivatives move you upward. Integrals move you downward.

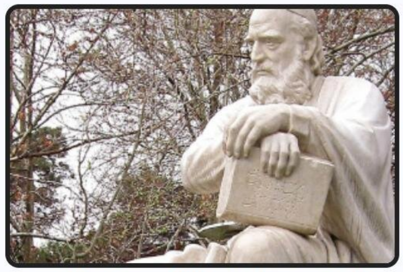
This activity is their first real encounter with the **Fundamental Theorem of Calculus**, even if we do not yet name it formally.

📘 Adventure 6 Story – Galileo and the Secret of Motion

How adding small pieces revealed the laws of nature



Galileo Galilei



Omar Khayyam - Jalali Calendar - quatrains (Robayat)

Galileo before the Holy Office,

In the early 1600s, when most people believed the universe was already understood, one stubborn Italian scientist refused to accept answers based on tradition alone. His name was **Galileo Galilei**.

Galileo wanted evidence. One question especially bothered him: **What really happens when something speeds up?** People could measure how fast an object moved at a moment, or how far it traveled overall—but no one understood how **distance builds up when speed keeps changing**. That missing idea lies at the heart of calculus.

Slowing the World Down: Falling objects moved too fast to study directly, so Galileo did something clever. He slowed motion down. He built a long wooden ramp—an **inclined plane**—with a smooth groove carved into it. A bronze ball rolled down the ramp slowly enough to observe carefully. To measure time, Galileo used a **water clock**, letting water drip steadily into a container.

At equal time intervals, he marked how far the ball had traveled. **What he found was revolutionary: Speed increased steadily; Distance did not increase evenly; Total distance came from adding many small pieces of motion.**

Galileo did not use integrals—they did not exist yet. But conceptually, he was already doing the same thing students do when they add rectangles under a velocity curve.

Motion as Accumulation. Galileo realized something profound: **Distance is not determined by speed at one moment, but by how speed accumulates over time.** In modern language, we say: **Distance is the area under the velocity curve.**

When students estimate distance using larger and then smaller time intervals, and watch their answers converge, they are repeating Galileo's method—only with graphs instead of ramps.

Looking to the Sky - Galileo's curiosity extended beyond motion on Earth. When he turned a telescope toward the sky, he observed moons orbiting Jupiter, mountains on the Moon, and the phases of Venus.

A thousand years ago, Persian scholars were among the world's best observers of the sky. Astronomers like Omar Khayyam measured the Sun's motion so precisely that the Jalali Calendar he helped design remains one of the most accurate calendars ever created. Their observations helped transform the sky from myth into mathematics.

These discoveries matched the precise planetary data collected earlier by **Tycho Brahe**, a Danish astronomer who spent decades making the most accurate measurements ever recorded. After Tycho's death, his assistant **Johannes Kepler** used that data to discover the true laws of planetary motion—showing that planets move in ellipses and change speed as they orbit.

Kepler provided mathematical laws. Galileo provided physical understanding.

Together, they showed that nature follows patterns that can be measured, accumulated, and understood.

Conflict with Authority - Galileo's ideas challenged more than science. They challenged authority.

Church officials feared that saying Earth moved around the Sun would disrupt long-held beliefs and social order. In 1633, Galileo was forced to recant publicly and spent the rest of his life under house arrest. But evidence could not be undone.

Why Galileo Matters Here - Galileo taught the world a lasting lesson: **Big changes come from adding many small ones.** That idea connects distance, velocity, and acceleration—and leads directly to calculus.

When you add areas under a curve in this activity, you are not just doing mathematics.
You are finishing a discovery, Galileo began—one small piece at a time.

✅ Adventure 6 – Student Activity Answer Sheet Solutions

Solutions for estimating the area under the velocity function:

Rectangle Approximation (dt = 1)

Time interval	Velocity	Area of Rectangle
0–1	0	0
1–2	7	7
2–3	12	12
3–4	15	15
4–5	16	16
5–6	15	15
6–7	12	12
7–8	7	7
Total		84

Comparison of Estimates

Method	dt	Distance (m)
Coarse Rectangles	1.0	84.00
Medium Rectangles	0.5	85.00
Fine Rectangles	0.25	85.25
Exact Integral	—	85.33

Velocity Function

$$v(t) = 8t - t^2$$

Distance Function

$$x(t) = \int (8t - t^2)\,dt = 4t^2 - \frac{1}{3}t^3$$

Acceleration

$$a(t) = \frac{d}{dt}v(t) = 8 - 2t$$

Distance in First 2 Seconds

$$x(2) = 4(2)^2 - \frac{1}{3}(2)^3 = 16 - \frac{8}{3} = \frac{40}{3} \approx 13.33 \text{ m}$$

From the Chart 13

Distance from 2 to 7 Seconds

$$x(7) - x(2) = \left(4(49) - \frac{343}{3}\right) - \left(16 - \frac{8}{3}\right) = \frac{205}{3} \approx 68.33 \text{ m}$$

From the chart 82 - 13 = 69

Total Distance (0 to 8 seconds)

$$x(8) = \frac{256}{3} \approx 85.33 \text{ m}$$

Velocity after 8 seconds as the area under the acceleration curve

The velocity is 0 and the areas under the acceleration function are the sum of two triangles with areas 4 and -4 so it is zero.

Velocity change from 0 to 5 seconds

From the velocity curve: 16 -1 =15

From acceleration we get the same thing as areas of two triangles from 0-4 (16) and 4-5(-1)

Calculus

$x(t) = 4t^2 - t^3/3$ **d(x)/dt =** $8t - t^2$

$v(t) = t(8 - t) = 8t - t^2$ **d(v)/dt =** $8 - 2t$

$a(t) = 8 - 2t$

$x(t) =$ **∫v(t)dt = ∫(8t-**t^2**)dt = ∫** $8tdt - \int t^2$**dt =** $4t^2 - t^3/3$

$v(t) =$ **∫a(t)dt = ∫(8-2t)dt = ∫8dt-∫2t)dt =** $8t - t^2$

$a(t) = 8 - 2t$

Step	**Tip**
dt = 1 table	Let students compute areas manually — this anchors the meaning of area.
Summary table	Ask: “What do you notice as dt gets smaller?” Guide them to say *converges*.
Distance formula check	Have students compare graph read-off vs formula value to confirm consistency.
Braking phase	Clarify: **negative acceleration reduces velocity**, not distance.
Final reflection	Emphasize: **Area accumulates change.** This is the conceptual entry to integrals.

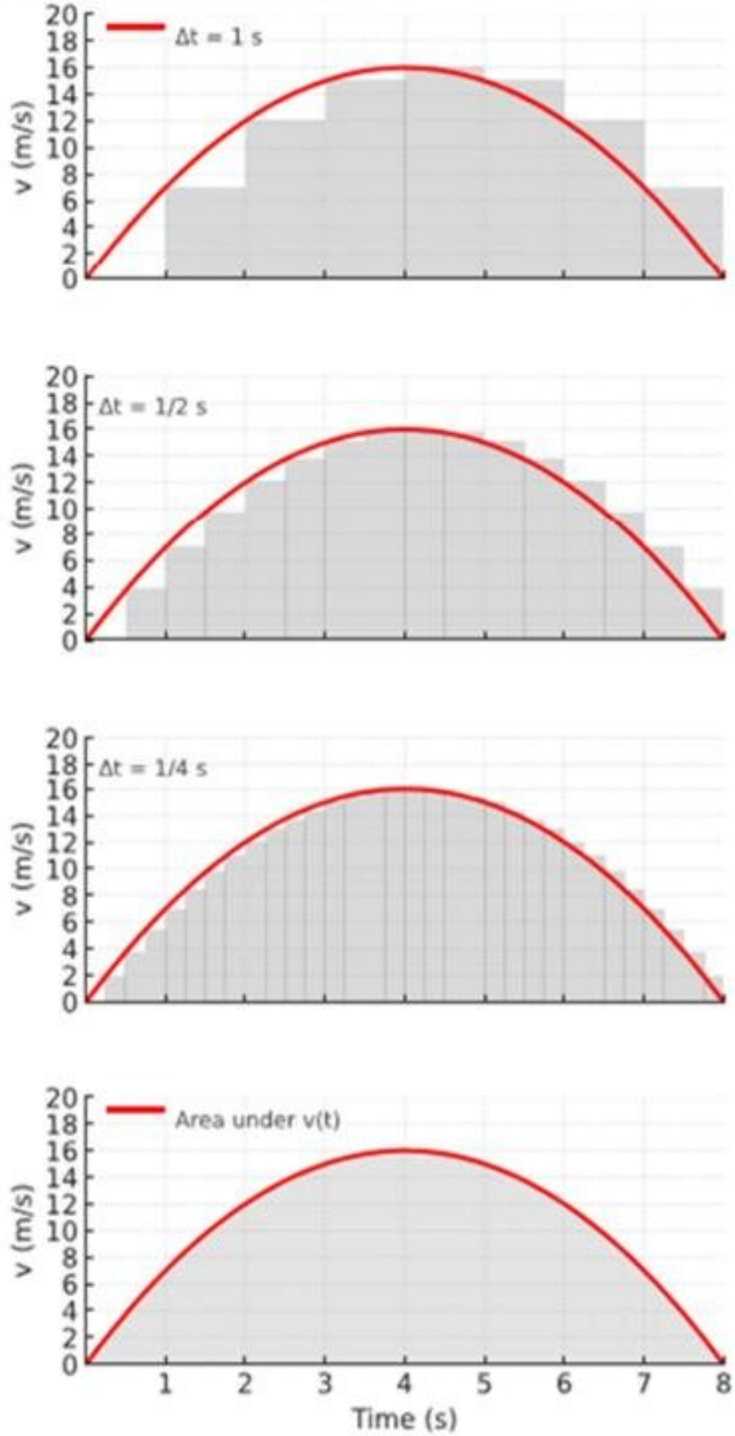
📘 Adventure 6 – Student Activity Worksheets

Student Exploration of Area under the Velocity Curve and Distance Traveled (0–8 s)

In this activity, you will estimate total distance by finding the area under the velocity curve.
1) Use the first table ($\Delta t = 1$ s). Look up velocity values from your v(t) chart, then compute the area of each rectangle ($v \times \Delta t$).
2) Compare your total with the sums for $\Delta t = 1/2$ s and $\Delta t = 1/4$ s.
3) Finally, compare all estimates with the exact integral value shown at the bottom.

Interval	Width (dt)	Velocity (lookup)	v × dt
0–1	1		
1–2	1		
2–3	1		
3–4	1		
4–5	1		
5–6	1		
6–7	1		
7–8	1		
Total (≈ m)			

Width (dt)	Method	Sum of Areas
1	Your Estimate	
1/2	Finer estimate	85.25
1/4	Very fine estimate	85.31



$$S = \int_0^8 v(t)\,dt = \int_0^8 (8t - t^2)\,dt$$
$$= [\,4t^2 - \tfrac{1}{3}t^3\,]_0^8$$
$$= (4 \cdot 8^2 - \tfrac{1}{3} \cdot 8^3) - 0$$
$$= 256 - \tfrac{512}{3} = 85\tfrac{1}{3}\text{ m} \approx 85.33\text{ m}$$

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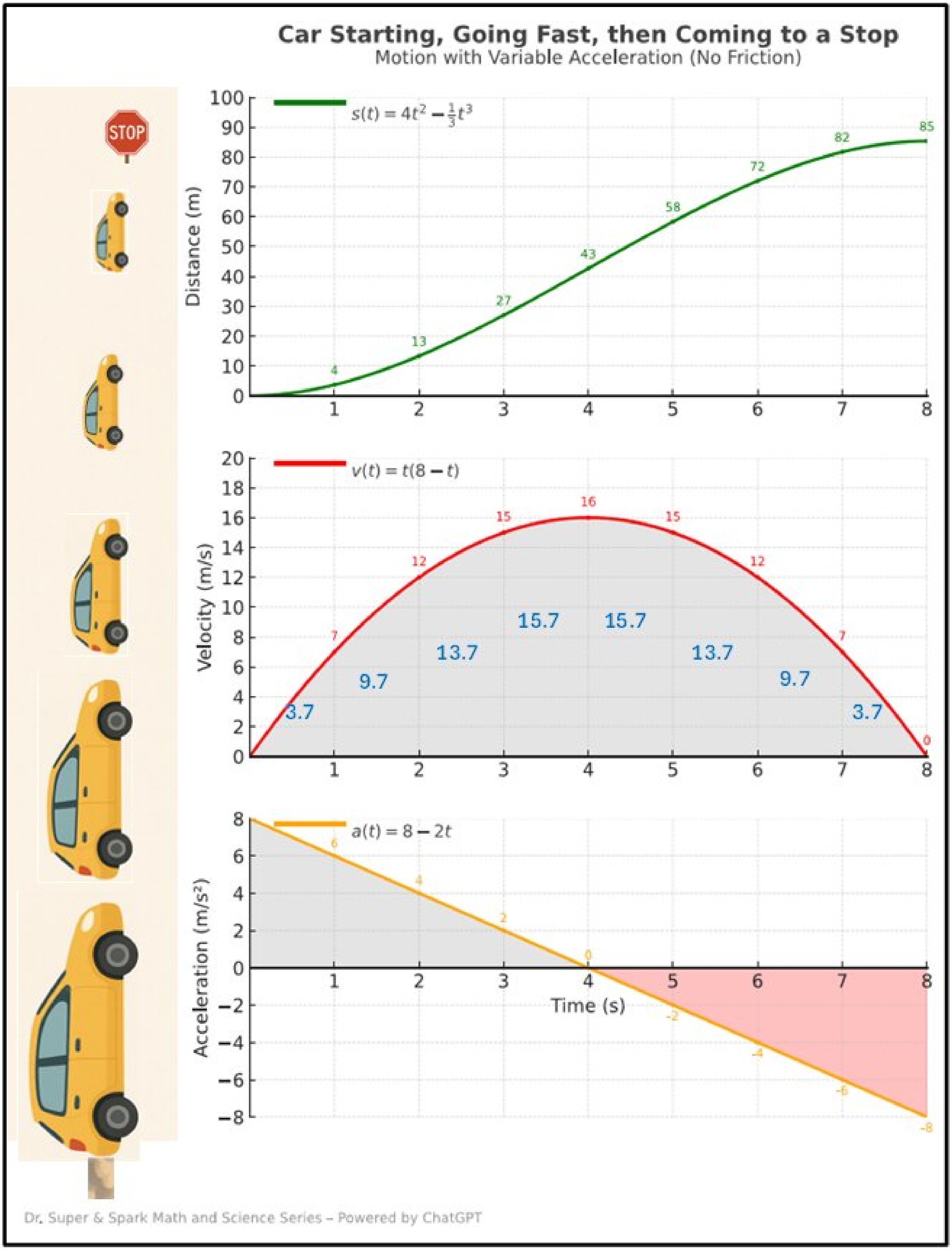
Car Starting, Going Fast, then Coming to a Stop
Motion with Variable Acceleration (No Friction)
STOP
$s(t) = 4t^2 - \frac{1}{3}t^3$
Distance (m)
4
13
27
43
58
72
82
85
$v(t) = t(8 - t)$
Velocity (m/s)
7
12
15
16
15
12
7
0
3.7
9.7
13.7
15.7
15.7
13.7
9.7
3.7
$a(t) = 8 - 2t$
Acceleration (m/s²)
Time (s)
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📘 Adventure 6 – Student Worksheets Area under the Velocity Curve and Distance Traveled

Variable acceleration increases then reverses, no friction from rest - Time ticks every 1 second

✳ Equations of Motion

- $x(t) = 4t^2 - t^3/3$
- $v(t) = t(8 - t) = 8t - t^2$ **Antiderivative (integral) of 8t-t^2 = $4t^2 - t^3/3$**
- $a(t) = 8 - 2t$ **Antiderivative (integral) of 8-2t = $8t - t^2$**

🧩 Questions

1 How far, how fast?

Read the distance traveled at 2 seconds and distance traveled between 2 and 7 seconds from the distance vs time graph?

Distance traveled in 2 seconds: ____________

Distance traveled from 2 to 7 seconds: ____________

2.Area means distance

Evaluate the integral for the v(t) function to find the distance traveled in 2 seconds and between 2 and 7 seconds. You can alternatively calculate these values from the blue values on the on the Velocity chart.
Distance traveled in 2 seconds:

$$\int_0^2 (8t - t^2)dt = [4t^2 - t^3/3]^2 - [4t^2 - t^3/3]^0 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_-\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Distance traveled between 2 and 7 seconds:

$$\int_2^7 (8t - t^2)dt = [4t^2 - t^3/3]^7 - [4t^2 - t^3/3]^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_-\_\_\_\_$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Are these areas equal or close to what you found in part 1 of the exercise?

Yes **No**

3. Velocity as the Area Under the Acceleration Function

Find the area under the acceleration function from 0 to 8 seconds.

Remember that the area under the time axis is negative:

Area under Acceleration Function from 1 to 8 Seconds= Area From 1 to 4 from 4 to 8

=____________-____________

= ________________________

Velocity after 8 seconds from the Velocity Chart = ________________________

Are these two numbers equal? **Yes** **No**

4 Acceleration check

The acceleration graph is a straight line with negative slope. What does the rectangle's area from 0–5 tell you about the change in velocity?

What is the change in velocity from 0 to 5 seconds: ________________________

Is this equal to what you see on the velocity chart? **Yes** **No:**

5 (***) Calculus

Take derivatives to show that you can find the velocity and acceleration functions from distance and velocity functions.

$\boldsymbol{x(t) = 4t^2 - t^3/3}$ **d(x)/dt =** __________________

$\boldsymbol{v(t) = t(8-t) = 8t - t^2}$ **d(v)/dt =** ___________________

$\boldsymbol{a(t) = 8 - 2t}$

Take the antiderivative (integral) of acceleration and velocity functions to get the velocity and distance functions.

$\boldsymbol{x(t) = 4t^2 - t^3/3}$

∫v(t)dt = ∫(8t-t^2)dt = ∫ $8tdt - \int t^2$dt

$\boldsymbol{v(t) = t(8-t) = 8t - t^2}$ **=** ____

$\boldsymbol{a(t) = 8 - 2t}$ **∫a(t)dt = ∫(8-2t)dt = ∫8dt-∫2t)dt =** ______________

🧑‍🏫 Adventure 7 – Momentum, Force & the Rocket Equation

🎯 Purpose of This Activity

Students explore the relationship between **mass**, **velocity**, **momentum**, and **force** in the context of a **simplified 60-minute rocket launch profile**. The goal is to understand that:

1. **Momentum is always** $p(t) = m(t)v(t)$.
2. **Force is the rate of change of momentum**,

$$F(t) = \frac{dp}{dt}.$$

For a rocket with decreasing mass,

$$F(t) = m(t)a(t) + v(t)\frac{dm}{dt}.$$

The second term is negative because $dm/dt < 0$.

3. Real rockets experience thrust *only* from fuel ejection.

🎬 Opening Narrative (Teacher Reads Aloud)

Before beginning, read your **Rocket & Stargazing Story**, which connects:

- ancient astronomy
- the wonder of looking backward in time through starlight
- curiosity about the cosmos
- rockets as the modern expression of the same desire
- and calculus as the language that describes how things move.

This sets the emotional stage.

🎥 Videos to Show Before the Activity

1️⃣ 3Blue1Brown — "Visualizing the Chain Rule"

Link: https://www.youtube.com/watch?v=YG15m2VwSjA
This is Video #4 in the Essence of Calculus series.
Students will see how **products of changing quantities** naturally lead to the chain rule and product rule.
This is the *conceptual anchor* for why the rocket equation has two pieces.

2️⃣ Falcon Heavy Europa Clipper Launch

Link: https://www.youtube.com/watch?v=eGGEh_Y82qM
Play the first 60–90 seconds.
Ask students:

- When does the rocket accelerate fastest?
- When does it coast?
- What is happening to the rocket when it is coasting?
- When does second-stage ignition occur?
- How does its mass change throughout launch?

This prepares them to interpret the PMCT charts.

📊 Charts Used in This Activity

You will use **two sets of charts**:

Chart A — Distance, Velocity, Acceleration (D–V–A)

Shows rocket motion during the first **60 minutes** of flight:

- **Stage 1 (0–10 min):** Accelerating upward
- **Stage 2 (10–50):** Coasting at nearly constant velocity
- **Stage 3 (50–60):** Final push toward escape speed

Chart B — Mass, Momentum, Force (M–P–F)

Piecewise:

- **Mass decreases** quickly in Stage 1
- **Mass constant** in Stage 2
- **Mass decreases slowly** in Stage 3
- Momentum and force follow from the calculus relationships.

🧠 Teacher Background Explanation

1. Momentum

$$p(t) = m(t)v(t)$$

2. Force is the derivative of momentum

$$F(t) = \frac{dp}{dt}.$$

3. When both mass and velocity change

Apply the **product rule**:

$$\frac{d}{dt}[m(t)v(t)] = m(t)a(t) + v(t)\frac{dm}{dt}.$$

4. Understanding the two terms

- $m(t)a(t)$
- The rocket's acceleration acting on its current mass.
  This is **not** the cause of thrust — it is the *effect* of thrust.
- $v(t)\frac{dm}{dt}$
- This comes from the rocket *losing mass*.
- Since $dm/dt < 0$, this term is negative and reduces the value of F.

5. Thrust vs. the formula

The rocket's *true engine thrust* depends on **exhaust velocity**, NOT the rocket's velocity.

We avoid exhaust velocity for now — too many variables — and use the simpler formula suitable for this curriculum.

This aligns perfectly with the PMCT charts.

🔍 Guiding Students Through the Activity Sheet

Part 1 — Reading the D–V–A Chart

Students identify:

- where acceleration is positive, zero, or negative
- where velocity is constant
- where distance is increasing slowly vs. rapidly
- how slopes, curvature, and areas relate

Part 2 — Interpreting Mass–Momentum–Force

Students:

- read mass at specific times
- compute momentum
- compute force using

$$F(t) = m(t)a(t) + v(t)\frac{dm}{dt}$$

compare their computed values with chart labels

Part 3 — Linking Charts A & B

Ask:

- "Why does force drop to zero during coasting?"
- "Why does momentum keep rising even when force is zero?"
- "What happens when mass changes?"
- "Is it possible for force to be small even when velocity is large?"
- "Which stage uses the most fuel? How do we know?"

💭 Questions for Class Discussion

1. *What causes a rocket to accelerate?*
2. **Fuel ejection**, not "pushing against air."
3. *Why do we need the product rule?*
4. Because **both** mass and velocity change.
5. *Why can force be zero even when velocity is huge?*
6. Because momentum is constant (no external forces).
7. *Why does the mass chart matter?*
8. It determines the sign and size of $v(t)\frac{dm}{dt}$.
9. *What if mass were constant?* Then

$$F = ma$$

The simple version of Newtons Second law

🧩 Extension or Challenge Problems (Optional)

Try:

- Write a simplified mass function $m(t)$for Stage 1 and Stage 3.
- Predict when coasting begins using only the mass chart.
- Compare rocket flight to car motion — what is similar, what is not?
- What happens if fuel is ejected twice as fast?

🪐 Closing Note for Teachers

This activity unifies **derivatives**, **products**, **chain rule**, **physics**, **momentum**, and **real rocket launch behavior**

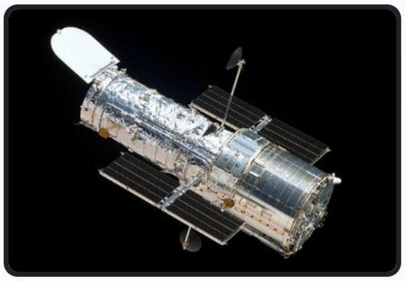
Students don't need the full Tsiolkovsky rocket equation — just the core idea:

Momentum changes lead to force, and mass loss changes momentum.

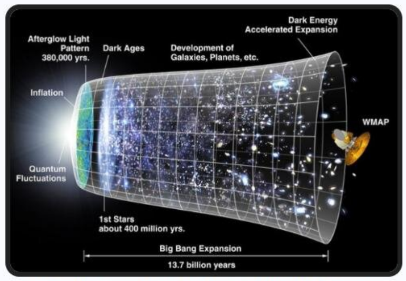
The videos (3Blue1Brown + Falcon Heavy launch) make this *alive*, and your storytelling gives emotional context that helps them remember.

Adventure 7 Story – A Short History of Stargazing and Space Travel

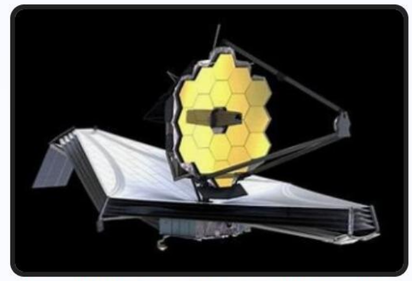
From Ancient Wonder to Modern Rockets and the Edge of Time



Hubble Space Telescope - 1990 to Present



Timeline and expansion of the universe



James Webb Space Telescope (JWST) - 2021 to present

For thousands of years, people looked up at the night sky and believed they were seeing the stars exactly as they were in that moment. They drew constellations, told stories, predicted seasons, and navigated oceans by those tiny lights. The sky felt eternal and unchanging.

What ancient people didn't know is one of the most astonishing ideas in science: when we look at the stars, we are looking into the past. Light takes time to travel. A star 100 light-years away shows us what it looked like 100 years ago. A galaxy a million light-years away shows us a million years into the past.

Only in the last century have we built tools powerful enough to truly understand this. The Hubble Space Telescope revealed galaxies so far away that their light began its journey when dinosaurs walked the Earth. And today, the James Webb Space Telescope can see even farther — capturing light from the first galaxies that formed after the Big Bang. Seeing the early universe is a modern superpower that didn't exist for 99.99% of human history.

A thousand years ago, Persian scholars were among the world's best observers of the sky. Astronomers like Omar Khayyam measured the Sun's motion so precisely that the Jalali Calendar he helped design remains one of the most accurate calendars ever created. Their observations helped transform the sky from myth into mathematics.

Fast forward to the 20th century: rockets finally carried us closer to the stars. Before humans flew, scientists sent animals — monkeys, dogs, and chimpanzees — to learn how living bodies react in space. Then, in 1961, Yuri Gagarin became the first human to orbit Earth, calling out, "I see Earth… it is beautiful." After that the space race started and eight years later, humans walked on the Moon for the very first time.

Today, rockets like SpaceX's Falcon Heavy and Starship launch spacecraft toward planets, asteroids, and icy moons. Future missions will explore worlds like Europa, a moon hiding a massive ocean beneath its frozen surface — one of the best places to search for life.

From ancient stargazers to Persian astronomers, to modern rocket scientists, one thing has always stayed the same: Humans look up. We wonder. And we explore. And the next chapter belongs to students like you.

Adventure 7 - Student Activity Worksheets Solutions

Rocket Launch: Momentum, Velocity, and Force

EQUATIONS OF MOTION

Mass (kg):

$m(t) = 50{,}000 - 2{,}000t$ for $0 \le t \le 10$ min

$m(t) = 30{,}000$ for $10 \le t \le 55$ min

$m(t) = 30{,}000 - 600(t-55)$ for $55 \le t \le 60$ min

Velocity (km/s):

v(t) is read from the Rocket Velocity Chart (red curve).

Connecting Mass, Velocity, Momentum, and Force

When a rocket launches, two things happen at the same time:

1. The rocket gains velocity (it speeds up).
2. The rocket loses mass (burning fuel).

Because momentum depends on BOTH of these:

$$p(t) = m(t) \cdot v(t)$$

The rate of change of momentum uses the Product Rule:

$$dp/dt = m(t)\, dv/dt + v(t)\, dm/dt$$

And force is defined as:

$$F(t) = dp/dt$$

$$F(t)=m(t)\, a(t) + v(t)\, dm/dt$$

This explains why force depends on BOTH acceleration and mass loss.

(Important Note) In the following tables notice that velocity from the chart is in km/sec and you need to convert to m/sec. Also, mass change dm/dt is in Kg/minute and you need to convert it to Kg/sec when you are calculating the force.

1. Estimate Velocity from the Slope of the Distance Curve and the Velocity Chart

Time (min)	Velocity =(dx/dt) km/sec	From Chart km/sec	From Chart m/sec
5	1000/5x60≈3.3	3.9	=3.9x1000=3,900
30	(21,000-2,000)/40x60≈7.9	7.8	=7.9x1000=7,800
58	(27000-24,00)/5x60=3,000/300≈10	11	=11x1000=11000

2. Estimate Acceleration from Slope of Velocity Curve and the Acceleration Chart

Time (min)	Acceleration a(t) ≈ dv/dt (m/s^2)	Acceleration from Chart (m/s^2)
5	=7.8x1000/10x60=78/6=13	13
30	=0	0
58	=(11.2-7.8)x1000/10x60=34/6=5.66	5.66

3. Determine Mass from the Mass Function

Time (min)	Mass (kg)
5	50,000-2000t=50,000-10,000=40,000
30	30,000
58	30,000 − 600(t−55) =30,000-3,000=27,000

4. Compute Mass Change (What is Pushing the Rocket) dm/dt

Time (min)	dm/dt (kg/min)	dm/dt (kg/sec)
5	d/dt[(50,000-2,000t)]=-2000	=-2000/60=33.3
30	d/dt[30,000)]=0	=0
58	d/dt[(30,000-600(t-55)]=-600	=-600/60=10

5. Compute Force F(t) = dp/dt= m(t) a(t) + v(t) dm/dt

Time (min)	m(t)a(t) (N)	dm/dt kg/sec	v(t)dm/dt (N)	Force F(t) (N)
5	=13x40,000=520,000	33.3	=3900x33.3=129,870	=520000-129870 =390,130
30	0	0	=7,.800x0=0	=0
58	=5.66x27,000=152,820	10	=11,000x10=110,000	=152820-110,000 =42,820

Verify that your answers for F(t) are close to what is shown on the Force chart.

Interpretation Questions

• When is the force largest? Why?

A: At the start of the launch of the rocket.

• Why is the force close to zero during the coasting phase?

A: At this point the rocket is not burning any fuel and just circling the earth

• Why does the force increase again near 58 minutes?

A: This is the sling shot effect just a small push to get out of the Earth orbit

- **. Actual Formulas for Distance, Velocity and Acceleration for our Charts**

Time (min)	**Distance (km)**	**Velocity (km/s)**	**Acceleration (m/ s^2)**
5	$0.0234t^2$	$0.78t$	13
30	$2.34 + 0.468(t - 10)$	7.8	0
58	$21.06 + 0.468(t - 50) + 0.0102(t - 50)^2$	$7.8 + 0.34(t - 50)$	5.66

- (***) Examining these actual formulas in the above table explain why the estimates for acceleration as slope of the velocity is an exact match but the estimates for velocity as slope of the distance curve is not an exact match.

• Why are the estimates for velocity as the slope of the distance curve not identical to the value on the velocity chart?

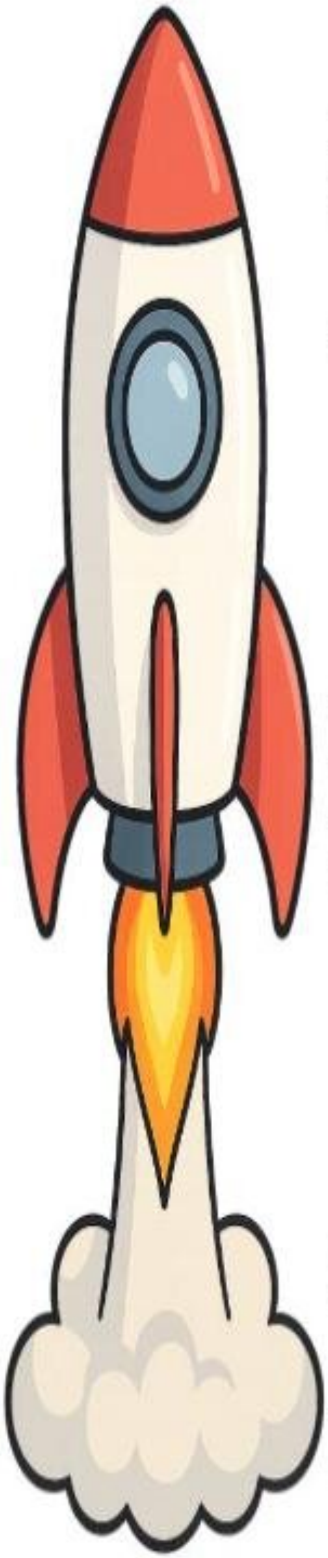
A: The distance curve is not a straight line (it is a quadratic) so we cannot get an exact rise over run figure

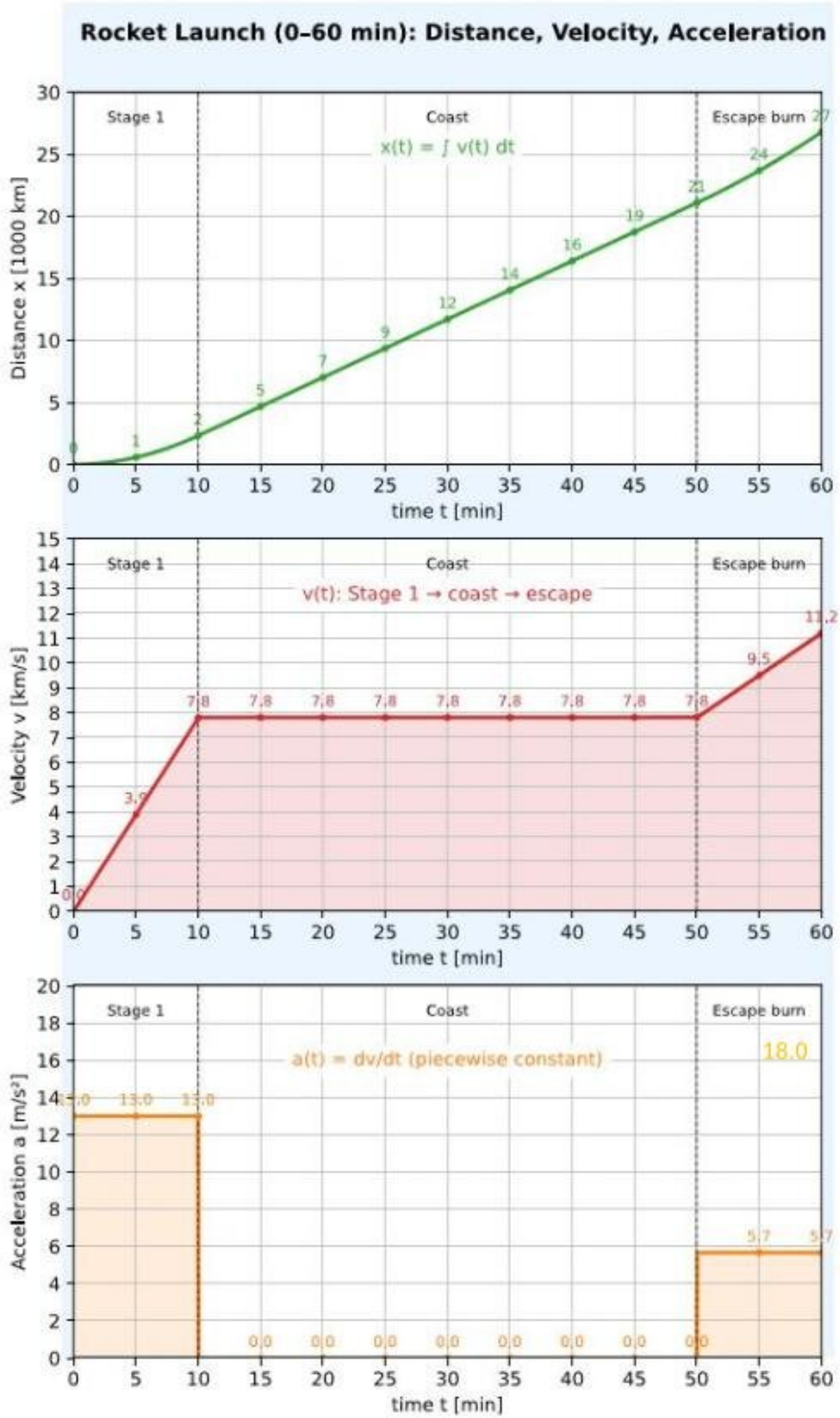
• Why are the estimates for acceleration as the slope of the velocity curve identical to the value on the acceleration chart?

A: Velocity curve is piecewise (linear) so the calculation of slope as rise over run is exact.

What makes the rocket go forward?

A: The fuel that is burned and is ejected from the rocket. It pushes the rocket and the same time makes the rocket become lighter and gain even more speed.





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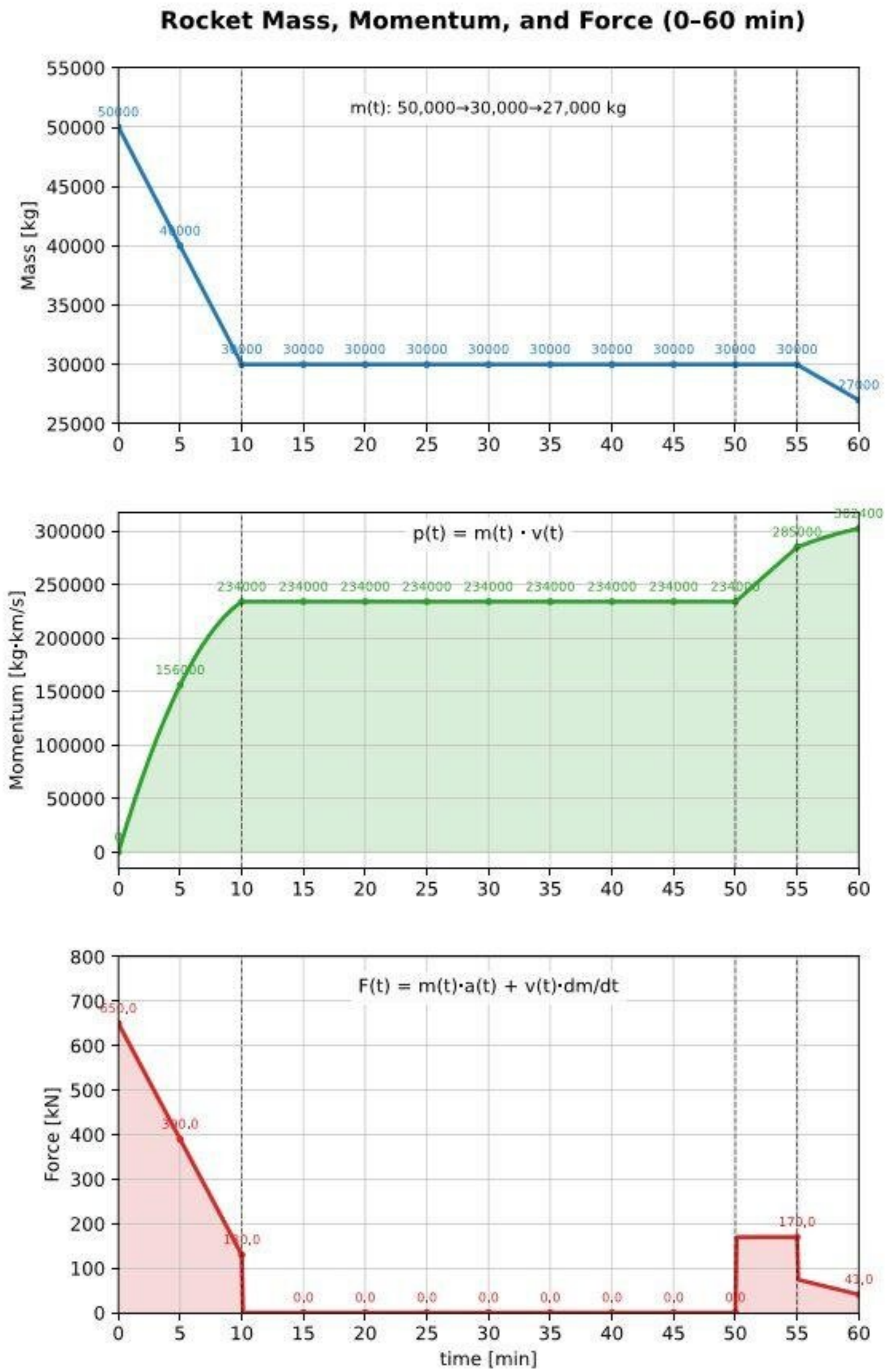
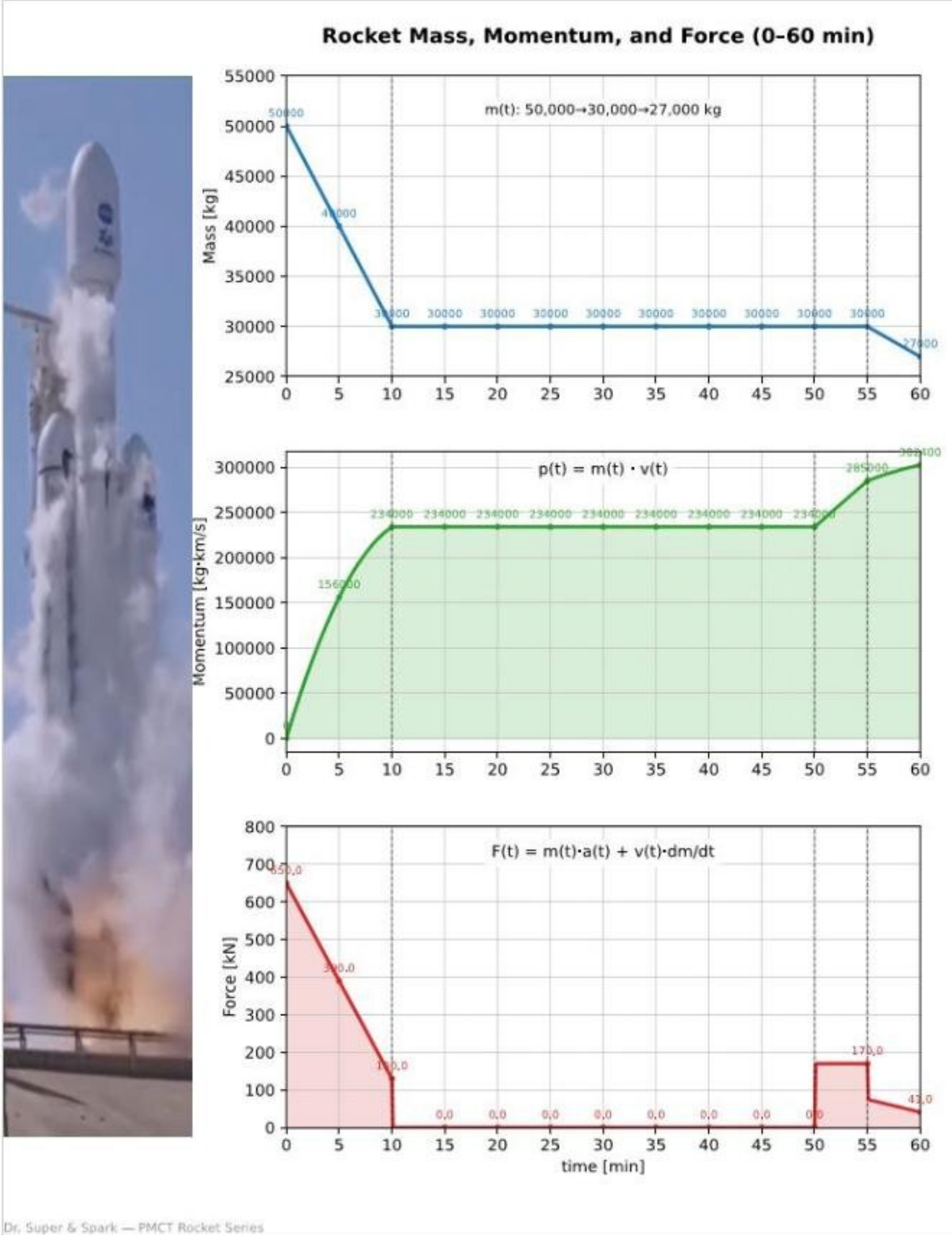


Dr. Super & Spark — PMCT Rocket Series


Rocket Mass, Momentum, and Force (0–60 min)
m(t): 50,000→30,000→27,000 kg
Mass [kg]
p(t) = m(t) · v(t)
Momentum [kg·km/s]
F(t) = m(t)·a(t) + v(t)·dm/dt
Force [kN]
time [min]
Dr. Super & Spark — PMCT Rocket Series

Adventure 7 – Student Activity Sheet

Rocket Launch: Momentum, Velocity, and Force

EQUATION OF MOTION

Mass (kg):

$m(t) = 50{,}000 - 2{,}000t$ for $0 \le t \le 10$ min

$m(t) = 30{,}000$ for $10 \le t \le 55$ min

$m(t) = 30{,}000 - 600(t-55)$ for $55 \le t \le 60$ min

Velocity (km/s):

$v(t)$ is read from the Rocket Velocity Chart (red curve).

Connecting Mass, Velocity, Momentum, and Force

When a rocket launches, two things happen at the same time:

1. The rocket gains velocity (it speeds up).

2. The rocket loses mass (burning fuel).

Because momentum depends on BOTH of these:

$$p(t) = m(t) \cdot v(t)$$

The rate of change of momentum uses the Product Rule:

$$dp/dt = m(t)\, dv/dt + v(t)\, dm/dt$$

And force is defined as:

$$F(t) = dp/dt$$

$$F(t)=m(t)\, a(t) + v(t)\, dm/dt$$

This explains why force depends on BOTH acceleration and mass loss.

(Important Note) In the following tables notice that velocity from the chart is in km/sec and you need to convert to m/sec. Also, mass change dm/dt is in Kg/minute and you need to convert it to Kg/sec when you are calculating the force.

1. Estimate Velocity from the Slope of the Distance Curve and the Velocity Chart

Time (min)	Velocity =(dx/dt) km/sec	From Chart km/sec	From Chart m/sec
5	1000/5x60≈	=	= x1000=
30	(21,000-2,000)/40x60≈	=	= x1000=
58	(27000-24,00)/5x60=≈	=	= x1000=

2. Estimate Acceleration from Slope of Velocity Curve and the Acceleration Chart

Time (min)	Acceleration a(t) ≈ dv/dt (m/s^2)	Acceleration from Chart (m/s^2)
5	=7.8x1000/10x60=	
30	=0	
58	=(11.2-7.8)x1000/10x60=	

3. Determine Mass from the Mass Function

Time (min)	Mass (kg)
5	50,000-2000t=
30	30,000
58	30,000 − 600(t−55) =

4. Compute Mass Change (What is Pushing the Rocket) dm/dt

Time (min)	dm/dt (kg/min)	dm/dt (kg/sec)
5	d/dt[(50,000-2,000t)]=	=- /60=
30	d/dt[30,000)]=	=0
58	d/dt[(30,000-600(t-55)]=	=- /60=

5. Compute Force F(t) = dp/dt= m(t) a(t) + v(t) dm/dt

Time (min)	m(t)a(t) (N)	dm/dt kg/sec	v(t)dm/dt (N)	Force F(t) (N)
5	=13x40,000=		==3.9x1000x33.3=	= - =
30	0	0	=7.8x0=	=
58	=5.66x27,000=		=11x1,000x10=	= - =

Verify that your answers for F(t) are close to what is shown on the Force chart.

Interpretation Questions

- **When is the force largest? Why?**

____

• **Why is the force close to zero during the coasting phase?**

____

____

• **Why does the force increase again near 58 minutes?**

____

____

. Actual Formulas for Distance, Velocity and Acceleration for our Charts

Time (min)	Distance (1000 km)	Velocity (km/s)	Acceleration (m/ s^2)
5	$0.0234t^2$	$0.78t$	13
30	$2.34 + 0.468(t - 10)$	7.8	0
58	$21.06 + 0.468(t - 50) + 0.0102(t - 50)^2$	$7.8 + 0.34(t - 50)$	5.66

(***) Examining these actual formulas in the above table explain why the estimates for acceleration as slope of the velocity is an exact match but the estimates for velocity as slope of the distance curve is not an exact match.

• **Why are the estimates for velocity as the slope of the distance curve not identical to the value on the velocity chart?**

____

• **Why are the estimates for acceleration as the slope of the velocity curve identical to the value on the acceleration chart?**

____

What makes the rocket go forward?

____

🧑‍🏫 Adventure 8 – Derivatives, Antiderivatives and the DiVA Charts

🧩 Purpose

Adventure 8 is a **consolidation Adventure**. It brings together the algebraic rules of differentiation and integration with the graphical interpretation of motion using the **DiVA Chart (Distance → Velocity → Acceleration)**. This Adventure prepares students for Adventure 8 and Adventure 9, where ratios and long-time behavior depend critically on rates of change.

🧱 Prerequisites

Students should already be comfortable with:

- The meaning of derivative as *rate of change* (Adventures 3–6)
- Basic integration as *accumulated change*
- Reading Distance–Velocity–Acceleration charts
- Interpreting slope and area graphically

No formal limit theory is required.

🌍 Story Connection

Begin by reading the one-page story about Walter Alvarez and the thin iridium layer. The key idea is that a **very thin layer can encode an entire history**. Adventure 7 uses this same metaphor: motion leaves behind a thin informational layer — the **DiVA chart**. Distance, velocity, and acceleration are not separate topics; they are stacked views of the same motion, just as the clay layer revealed the story of the dinosaur extinction. This framing helps students see the worksheets as a single investigation rather than disconnected exercises.

📚 Set-Up

You will need:

- Adventure 8 Student Derivatives & Antiderivatives Worksheet
- Adventure 8Student DiVA Chart Worksheet
- Completed solution sheets (for teacher reference only)
- Pencil, ruler; calculator optional

Have students read the story first, then proceed to the worksheets.

🧮 Guided Procedure

1. Derivatives & Antiderivatives Warm-Up

Students complete the derivative and antiderivative tables.

Teaching focus:

- Rule identification before computation
- Recognizing when to use Power vs. Chain or Product Rule
- Using differentiation to check antiderivatives

Ask students:

- “Which rule applies here?”
- “What is changing, and how fast?”
- “How can you check this result?”

2. Completing the DiVA Charts

Students are given one motion equation per chart and must reconstruct the other two.

Teaching focus:

- Moving up and down the DiVA structure using differentiation and integration
- Connecting algebraic results to graphical features

Ask students:

- “Which DiVA layer is given?”
- “Are we moving up or down the chart?”
- “Does the slope of this graph match your equation?”
- “Does the shaded area explain the numerical result?”

Encourage students to use the graphs as **evidence**, not decoration.

🪄 Key Teaching Insight

Every step **down** the DiVA chart simplifies the situation.
Every step **up** accumulates history.

This idea is essential preparation for Adventure 9, where students will see that **rates (derivatives) control long-term behavior** much more clearly than raw values.

🧠 Discussion & Reflection

Use questions such as:

- “Why can the same motion be described three different ways?”
- “Why are slopes and areas more informative than raw numbers?”
- “How is the DiVA chart like the iridium layer in the story?”

Students who can explain the motion verbally and graphically are ready to move on.

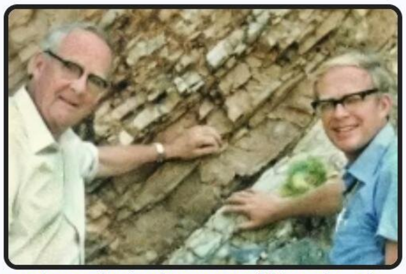
🪄 Teacher Sidebar for the story

Teacher Sidebar — Cultural Note on “DiVA”

The name DiVA comes from Distance–Velocity–Acceleration, but it also carries deep Indo-Iranian resonance. In early Indo-Iranian culture, Deva meant “shining one” — a guiding celestial power. During the evolution of early Iranian religion, especially through Mithraism and the teachings that became Zoroastrianism, the same root transformed into Div, meaning a chaotic or monstrous force. India kept the older meaning; Persia inverted it.

This polarity—Deva (light) ↔ Div (chaos)—is a powerful metaphor for physics: motion becomes clear and illuminated only when interpreted properly. When misunderstood, it seems chaotic and unpredictable. The DiVA Chart is presented as the guiding light that turns chaos into comprehension.

📘 Adventure 8 Story — The Layer That Changed the World



Walter and Luis Alvarez at Gubbio Italy - Big History on You Tube

the Bottaccione Gorge outcrop showing the cliff face where the K–Pg boundary layer is exposed.

Chicxulub crater - 66 million years ago when an asteroid, about ten kilometers in diameter, struck Earth.

The Dinosaur Extinction, the Missing Crater, and the DiVA Chart

In 1980, a young geologist named Walter Alvarez was climbing the limestone cliffs of Gubbio, Italy, when he noticed something odd: a very thin, chocolate-brown band of clay—only a couple of centimeters thick—squeezed between massive layers of limestone. It looked ordinary, but Walter's instincts said otherwise. He scraped a sample and took it home.

Walter's father, Luis Alvarez, a world-famous physicist and Nobel Prize winner, used to tease him: "Geology? That's not real science. That's just collecting rocks." But he respected curiosity—especially Walter's. So when he saw the strange clay, Luis contacted his colleague Frank Asaro, a nuclear chemist with one of the most advanced neutron-activation systems in the world. Asaro analyzed the sample using equipment normally reserved for high-precision physics.

The results were stunning: the clay contained an enormous spike of iridium—hundreds of times more than Earth's crust normally has. Iridium is extremely rare on Earth but common in asteroids.

Then came the breakthrough. Geologists began reporting the same thin layer in Denmark, Spain, New Zealand, Egypt, North America, and even deep-sea drill cores. A global stripe of iridium-rich clay, laid down everywhere at the same moment—the instant the dinosaurs vanished.

Walter, Luis, Asaro, and others pieced together the story: a six-mile-wide asteroid had slammed into Earth. Dust from the explosion, rich in iridium, circled the globe and settled into a slender stripe of clay—thin as a coin, yet recording one of history's greatest catastrophes.

One mystery remained: Where was the crater? For years, no one could find it. Then petroleum geologists in Mexico mapped unusual gravity readings under the Yucatán Peninsula. Their data revealed a buried circular scar, 110 miles wide. The Chicxulub crater—exactly the right age—had finally been found. The last missing piece clicked into place. And Luis Alvarez never again joked about Walter's "rock collecting."

The DiVA Chart: Motion's Iridium Layer: Just as the thin iridium layer reveals Earth's past, motion leaves behind its own thin stripe that reveals how something moved.

A vertical stack of three curves: Distance; Velocity; Acceleration

Together they form the DiVA Chart—the Distance–Velocity–Acceleration layer. The name DiVA echoes the ancient Indo-Iranian word Deva, meaning "shining one, guiding light." In later Persian tradition the same root became Div, a creature of chaos—reflecting a deep cultural inversion during Mithraic and proto-Zoroastrian transformations.

Motion behaves the same way: when understood, it is elegant and clear like a Deva; when misunderstood, it feels chaotic like a Div. The DiVA Chart is the guiding light that makes motion readable. Just as geologists learned to read Earth's past from a thin band of clay, you will learn to read motion from this thin band of calculus. Welcome to Adventure 8. Today you learn to read the layer.

Using the Sum, Product and Chain Rule Find the Derivative for Each Row

The Derivaxive of f(x) is witten as df(x)/dx or as f'(x)

Sum Rule: d(f(x)+g(x))/d(x)=d(f(x)/dx + d(g(x)/dx

Producx Rule: d(f(x)g(x))/dx = f'(x)g(x) + f(x)g'(x)

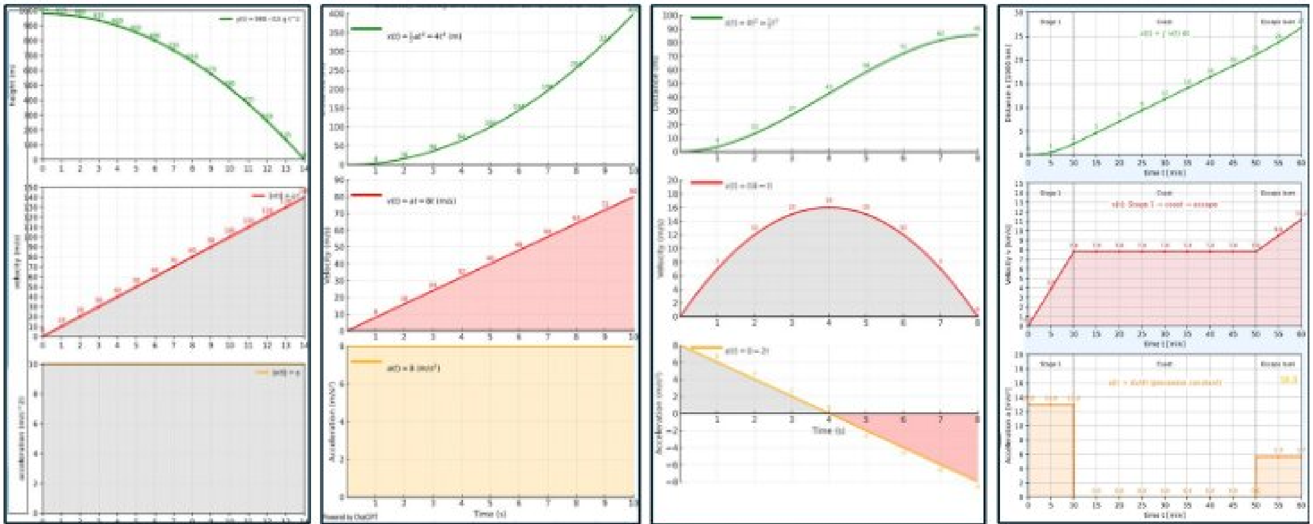
Chain Rule: d(f(g(x)))/dx = f'(g(x)) g'(x)

Function	Derivative
Sin(x)	cos(x)
cos(x)	-sin(x)
ln(x)	1/x
x^2+x^3	$2x+3x^2$
3ln(x)	3/x
e^{-kt}	$-ke^{-kt}$
x^2e^x	$2xe^x+x^2e^x = (x^2+2x)e^x$
$\sin(x)e^x$	$\cos(x)e^x+\sin(x)e^x=(\cos(x)+\sin(x))e^x$
$\sin(x^2)$	$2x\cos(x^2)$
*tan(x)=sin(x)/cos(x)	$\cos x/\cos(x)+\sin(x)\sin(x)/\cos^2(x)=1+\sin^2(x)/\cos^2(x)=1+\tan^2(x)$
xln(x)	ln(x)+x(1/x)=**ln(x)+1**
$(x-1)^2$	2(x-1)
(x-5)(x-3)	(x-3)+(x-5)=**2x+8**
e^{-x2}	$-2xe^{-x2}$

Using the Power Rule ***Find the Derivative or Antiderivative for Each Row***

The Derivative of f(x) is written as df(x)/dx or df/dx or f'(x) The Derivative of a constant is 0 The Antiderivative (Integral) of df(x)/d(x) is f(x)	
Function f(x)	**Derivative df(x/dx)**
ax^n	anx^{n-1}
7.8	0
$(13/2)x^2$	13x
$1/2gx^2$ (g=10)	10x
$4x^2-x^3/3$	$8x-x^2$
ax (a constant)	a (constant)
$(1/2)ax^2$	ax (a constant)
gx (g=10)	10
$5x^2$ (g=10)	gx (g=10)
$4x^2-x^3/3$	$8x-x^2$
$6.5x^2$	13x
ax (a constant)	a
13x	13
13x	13

Compute the 2 Missing Equations of Motion for each Set of Charts



$y(t) = 980 - \frac{1}{2}gt^2$ $(g = \frac{10m}{sec2})$

v(t)= **10t**

a(t)= 10

Find Height, Velocity, Accelaration after 7 seconds.

y(7)=980-5x49=**735**

v(7)=**70**

a(7)=**10**

$x(t) = \frac{1}{2}at^2$

v(t)=**at**

a(t)= a (constant)

Read **a** from the chart? *a*= **8**

Is the slope of v(t) equal to a?

Slope of v(t)=(v(10)-v(0)/10
= 80-0/10-0
= 80/10 = **8**

Yes No

---

x(t)= **4t²-(1/3)t³**

$v(t) = t(8 - t) = 8t - t^2$

a(t) = **8-2t**

Check that after 4 seconds the area under the accelaration is equal to v(4).

Area Under **a** up to 4 =4x8/2=**16**

v(4)= 16

Yes No

If x(3)=27 and x(5)=58.3

Estimate v(4)=rise/run
=(58.3-27)/2=**15.65**

Compare with v(4) = **16** (chart)

(0-10 minutes)

x(t)= **7.5t²**

v(t)=13t

a(t)= **13**

(10-50 minutes)

x(t)= 7.8t-2,340 (1,000 km)

v(t)= **7.8** **(Km/sec)**

a(t)= **0** **((Km/sec²)**

Find Distance and Velocity after 20 minutes (20x60=1,200 sec)

x(20 min)=**7.8x1200-2,340 =7,020 km**

v(20 min)=**1200x20=24,000 (Km/sec)**

a(20) = **0**

Compare with Chart Values

Agrees with Chart **Yes** No

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📘 Adventure 8 — Student Activity Sheets:

Derivatives, Antiderivatives & the DiVA Chart

🧩 Purpose (For Students)

In this activity you will review how derivatives and antiderivatives describe **change** and **accumulation**, and then use those ideas to complete **DiVA charts** (Distance–Velocity–Acceleration). Your goal is to learn how to *read motion as a story*, using equations, slopes, and areas.

🧮 Part 1 — Derivatives & Antiderivatives

Use the rules printed on the worksheet to complete the tables.

- Use the **Power Rule** for simple polynomials.
- Use the **Sum, Product, and Chain Rules** where needed.
- When asked for an **antiderivative**, reverse the power rule.

What to do:

- Fill in the missing derivatives or antiderivatives.
- Check your work by asking: *"If I differentiate this result, do I get back the original function?"*

📊 Part 2 — Completing the DiVA Charts

Each DiVA chart shows **one equation of motion**:

- Distance $x(t)$, **or**
- Velocity $v(t)$, **or**
- Acceleration $a(t)$

Your job is to find the **other two equations** and complete the questions at the bottom of each chart.

Remember the DiVA connections:

- Differentiate to go **down** the chart
  $x'(t) = v(t), \quad v'(t) = a(t)$
- Integrate to go **up** the chart: $\int a(t)\,dt = v(t), \quad \int v(t)\,dt = x(t)$

What to do for each chart set:

1. Write the two missing equations.
2. Answer the numerical questions at the bottom (values, slopes, or areas).
3. Use the graphs to check your answers:
   - Slopes ↔ derivatives
   - Shaded areas ↔ integrals

🧠 Final Check

If your equations, numbers, and graphs all agree, you are reading the DiVA chart correctly.

Using the Power Rule ***Find the Derivative or Antiderivative for Each Row***

The Derivative of f(x) is written as df(x)/dx or df/dx or f'(x) The Derivative of a constant is 0 The Antiderivative (Integral) of df(x)/d(x) is f(x)	
Function f(x)	**Derivative df(x/dx)**
ax^n	anx^{n-1}
7.8	
	13x
$1/2gx^2$ (g=10)	
	$8x-x^2$
	a (constant)
	ax (a constant)
gx (g=10)	
	gx (g=10)
$4x^2-x^3/3$	
$6.5x^2$	
ax (a constant)	
	13
13x	

Using the Sum, Product and Chain Rule Find the Derivative for Each Row

The Derivaxive of f(x) is witten as df(x)/dx or as f'(x)

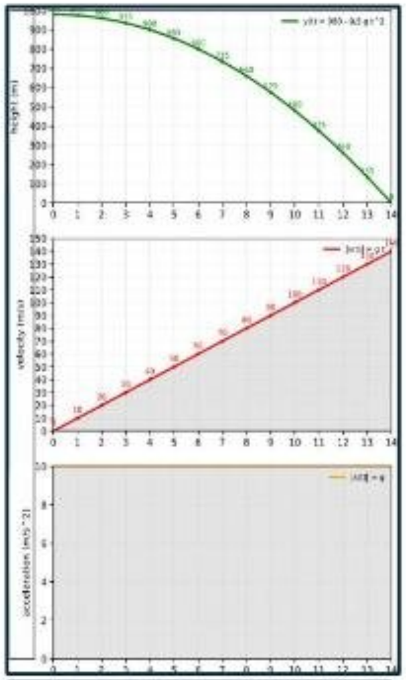
Sum Rule: d(f(x)+g(x))/d(x)=d(f(x)/dx + d(g(x)/dx

Producx Rule: d(f(x)g(x))/dx = f'(x)g(x) + f(x)g'(x)

Chain Rule: d(f(g(x)))/dx = f'(g(x)) g'(x)

Function	Derivative
Sin(x)	cos(x)
cos(x)	-sin(x)
ln(x)	1/x
x^2+x^3	
3ln(x)	
e^{-kt}	
x^2e^x	
$\sin x e^x$	
$\sin(x^2)$	
*tan(x)=sin(x)/cos(x)	
xln(x)	
$(x-1)^2$	
(x-5)(x-3)	
e^{-x2}	

Compute the 2 Missing Equations of Motion for each Set of Charts



$y(t) = 980 - \frac{1}{2}gt^2 \qquad (g = \frac{10m}{sec2})$

v(t)=

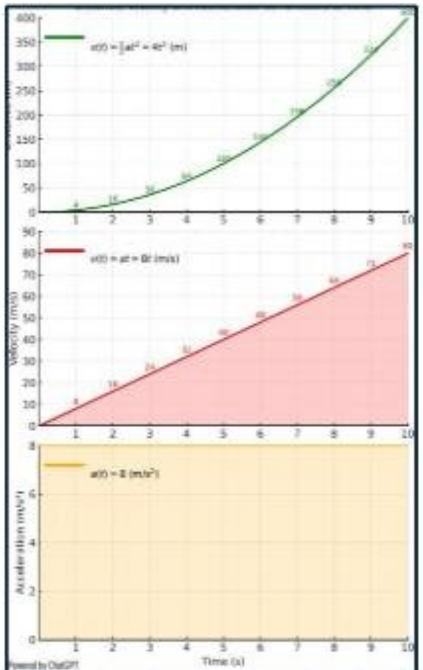
a(t)=

Find Height, Velocity, Accelaration after 7 seconds.

y(7)=

v(7)=

a(7)=



$x(t) =$ ____________________

v(t)=____________________

____

a(t)= a (constant)

____

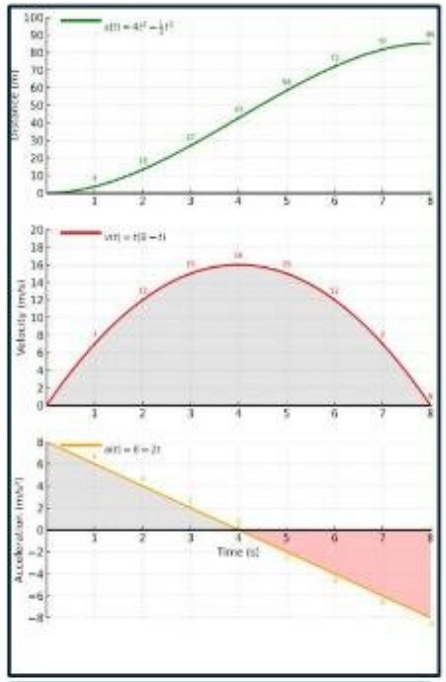
Read **a** from the chart? **a=**

Is the slope of v(t) equal to a?

Slope of v(t)=(v(10)-v(0)/10

=

Yes No



x(t)=

$v(t) = t(8 - t) = 8t - t^2$

a(t) =

Check that after 4 seconds the area under the accelaration is equal to v(4).

Area Under **a** up to 4 =

v(4)=

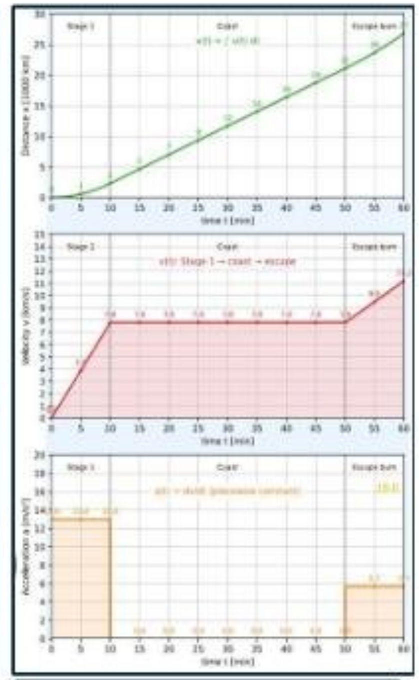
Yes No

If x(3)=27 and x(5)=58.3

Estimate v(4)=rise/run

=

Compare with v(4) = (chart)



(0-10 minutes)

x(t)=

v(t)=13t

a(t)=

____

(10-50 minutes)

x(t)= 7.8t-2,340 (1,000 km)

v(t)= (Km/sec)

a(t)=

Find Distance and Velocity after 20 minutes (20x60=1,200 sec)

x(20 min) =

v(20 min) =

Compare with Chart Values

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🧑‍🏫 Adventure 9 – The Age of the Earth — Clair Patterson & Exponential Decay

🧩 Purpose

This activity introduces students to **exponential decay**, **half-life**, and the astonishing story of how **Clair Patterson** used U-235 and U-238 to calculate the **age of the Earth**.

Students will:

- work with the exponential decay model $M(t) = M_0 e^{-kt}$,and L(t) = M(0)-M(t)
- M(t) is the amount U-235 remaining and L(t) is the amount lead at time t
- use real U-235 half-life (0.704 billion years) to compute the decay constant,
- understand why the simplified classroom model is $M(t) = 1024e^{-t}$,
- read & interpret the **DiVA triple chart** (Mass, Decay Speed, Change in Decay Speed),
- estimate Earth's age using:
    - a **half-life table** (powers of 2),
    - visual reading from the **mass curve**,
    - and a **single guided integral**,
- connect mathematics with scientific history, environmental policy, and Earth science.

This activity combines calculus, exponential functions, physics, and one of the most important scientific detective stories ever told.

🧱 Prerequisites

Students should know:

- the idea of exponential growth/decay (basic),
- how to read labeled axes,
- what a derivative represents (Section 6 & 7 preparation),
- the meaning of a half-life from science class.

No prior knowledge of logarithms is required — a minimal introduction is included.

📊 Set-up

You will need:

- **Clair Patterson story,**
- **U-235 DiVA chart sheet (Portrait 3-stack)**,
- **Adventure 9 Student Activity Sheet**,
- **ruler/pencil,**
- **calculator (maybe for a few LN computations).**

🎥 Watch: [What's so special about Euler's number e?](#)
[3Blue1Brown — Essence of Calculus, Chapter 5](#)

📖 1) Read / Listen: Story

Read or have the students read aloud the **Patterson story** first to set emotional and scientific context.

🧮 Guided Procedure

1. Warm-Up: The Magic of e^x

Students explore the Taylor series for e^x and discover:

$$\frac{d}{dx}(e^x) = e^x.$$

Teaching point:

- Exponential curves *keep their shape* when you take derivatives.
- This is why the **Mass**, **Decay Speed**, and **Acceleration** curves look identical.

This sets up the DiVA chart beautifully.

2. Mini-Logarithm Primer (Very Short)

Introduce only what is essential:

- Logs undo exponentials.
- $\ln\left(\frac{1}{2}\right) = -\ln 2.$
- Necessary for solving half-life equations.

No deeper log theory needed.

How ever you may mention that Logarithms may be called "Spitters" as they will Spit out the powers of the base number.

For example,for base ten log: $\log(1)=\log(10^0)=0$, $\log(10)=\log(10^1)=1$; $\log(10{,}000)=\log(10^4)=4$.

Also the 4 laws for logarithms.

$$\log(ab) = \log(a) + \log(b)$$
$$\log(a^n) = n\log(a)$$
$$\log_b(a) = 1/\log_a(b)$$
$$\log_b(a) = \log_c(a)/\log_c(b)$$

3. Deriving the Decay Constant From Half-Life

Students start with the general model:

$$M(t) = M_0 e^{-kt}$$

Use the half-life condition:

$$M(T_{1/2}) = \frac{1}{2}M_0, T_{1/2} = 0.704.$$

They compute:

$$e^{-0.704k} = \frac{1}{2} \Rightarrow k = \frac{\ln 2}{0.704} \approx 0.9846.$$

Teaching point:

- U-235 decays almost exactly like e^{-t}.
- This is why your simplified classroom model works so cleanly.

4. The Classroom Decay Model: $M(t) = 1024e^{-t}$

Explain:

- We begin with $M_0 = 1024$grams (students love powers of 2).
- Since $k \approx 1$, we use:

$$M(t) = 1024e^{-t}$$ **. and L(t) = M(0)-M(t) =1024 - $1024e^{-t}$**

This gives simple derivatives:

$$v(t) = \frac{dL(t)}{dt} = 1024e^{-t}, a(t) = 1024e^{-t}$$

All three curves on the DiVA chart have the same exponential shape. a(t) is the second derivative of M(t)

5. Reading the Triple Chart (DiVA)

Give time for visual exploration of the chart sheet.

Mass curves $M(t)$(green dashed) and L(t) (Green):

- Identify values at $t = 1, t = 2, t = 3$.
- Students see visually that U-235 shrinks rapidly at first, then slowly.
- Students see that as U-235 shrinks fast it is replaced by Lead

Decay speed $v(t)$(red):

- Ask: "What is happening to the *speed* of decay?"

Teaching point:
Exponential decay **slows down exponentially**.
Big idea: "More → faster decay. Less → slower decay."

Change in decay speed $a(t)$(orange):

- Same exponential shape.
- Reinforces how derivatives of exponentials behave.

Let students notice the repeated shape themselves.

(we have shown a(t) that is the second derivative of M(t) here for completeness)

6. Half-Life Table (Powers of Two)

Students fill the table:

1024 → 512 → 256 → 128 → 64 → 32 → 16 → 8

Key observation:

- 11 grams lies **between the 6th and 7th half-life**.

- So approximate age:

$$6.5 \times 0.704 \approx 4.58 \text{ billion years.}$$

The real age of Earth.

Let students own this discovery.

7. One Integral: Connecting Algebra to the Chart

The decay-speed curve

$$v(t) = 1024e^{-t}$$

tells how fast uranium is disappearing.

So the area under v(t) from 0 to 1 gives the total mass of U-235 lost:

$$L(1) = \int_0^1 v(t)\, dt.$$

Compute the loss:

$$L(1) = \int_0^1 1024\, e^{-t} dt \approx 647 \text{ grams.}$$

Convert this into uranium remaining – Use:

$$\mathrm{M}(1) = \mathrm{M}(0) - \mathrm{L}(1),$$

we get the Mass left after 1 half-life:

$$M(1) = 1024 - 647 = 377 \text{ grams.}$$

Teaching point:

- Their integral result **matches the value of mass converted to lead at L(1) curve at $t = 1$.**
- **The difference matches the value at M(1) the Uranium remaining**
- They see integrals as "area under a curve" in a real physical context.

8. Final Age-of-Earth Estimate from the Chart

Students find on the mass curve where:

$$M(t) \approx 11 \text{ g.}$$

They read $t \approx 4.5$–4.6billion years.

Two independent methods now agree:

- **Half-life table**
- **Visual reading from the chart**

This mirrors the real method Patterson used with meteorites.

🧠 Discussion & Reflection

Use these to deepen understanding:

- Why is exponential decay so smooth and predictable?

- Why does every half-life slice the remaining mass in half?
- Why do all three curves (M, v, a) have the same shape?
- What does the story of Clair Patterson show about:
  - scientific integrity,
  - environmental responsibility,
  - and careful measurement?
- How close were their estimates to the known value 4.54 billion years?

This reflection reinforces the scientific and moral dimensions of the story.

🧑‍🏫 Teacher Tips

- Shade the areas under the red and orange curves (students love visuals).
- Use the powers-of-2 pattern as an anchor for weaker students.
- Encourage estimation rather than exactness.
- Emphasize units (grams, billion years).
- Use the Patterson story to tie mathematics to real scientific and societal impact.
- Reassure students: logs appear only briefly and with full guidance.
- If time: discuss why U-235 is better than U-238 for this activity.

🧩 Optional Extensions

For advanced or curious students:

- Compare the exact model $M(t) = 1024e^{-0.985t}$ to the simplified $1024e^{-t}$.
- Compute one more integral, e.g. $\int_0^2 v(t)\,dt$.
- Introduce $M(t) = M_0(1/2)^{t/T_{1/2}}$.
- Discuss the role of U-238 in Patterson's full 1956 calculation.
- Show how meteorite samples gave the most accurate isotopic ratios.

🪄 Wrap-Up

By the end of this activity, students understand:

- **Exponential decay** is predictable and mathematical.
- **Half-life** behaves like a "built-in clock."
- The **age of Earth** can be estimated using simple powers of 2.
- **U-235** decays fast enough to reveal deep time.
- **e^x** and its derivatives control the shapes of natural processes.
- **Clair Patterson's courage** changed both science and public health.

This is one of the most elegant demonstrations in the entire Calculus & Mechanics series — blending math, physics, geology, chemistry, and ethics into a single unforgettable lesson.

You may wish to discuss some of the following that was in the video with the students:

⭐ Short Explanation: Why All Exponentials Are Built From e, and Why e^x Is Special

Every positive number N can be rewritten using the exponential e.
This comes from the fact that:

$$N = e^{\ln(N)}.$$

If we raise both sides to the power x, we get:

$$N^x = \left(e^{\ln(N)}\right)^x = e^{x\ln(N)}.$$

This works for **any** base a:

$$a^x = e^{x\ln(a)}.$$

And for powers of 2:

$$2^x = e^{x\ln(2)}.$$

⭐ Why the derivative of a^x is $\ln(a)\, a^x$

Start with the identity:

$$a^x = e^{x\ln(a)}.$$

Differentiate the right-hand side using the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}\left(e^{x\ln(a)}\right) = e^{x\ln(a)} \cdot \ln(a) = \ln(a)\ a^x.$$

So **every exponential grows at a rate proportional to itself**,
and the proportionality constant is $\ln(a)$.

⭐ Why e^x is the ONLY exponential equal to its own derivative

Plug $a = e$ into the formula:

$$\frac{d}{dx}e^x = \ln(e)\, e^x = 1 \cdot e^x = e^x.$$

So:

e^x is the only exponential function whose derivative is exactly the same as the original function.

This is why the base e is the natural one for calculus, physics, and especially radioactive decay.

⭐ Optional one-sentence version

Because $a^x = e^{x\ln(a)}$, its derivative is $\ln(a)\, a^x$.

Only when $a = e$ does $\ln(a) = 1$, making e^x the one exponential equal to its own derivative.

$$N = e^{\ln(N)}$$

gives back: $\ln(N) = \ln(N)$.

This is important for the kids because it shows *why* the identity $N = e^{\ln(N)}$ is true in the first place.

⭐ Why taking natural logs of $N = e^{\ln(N)}$ gives $\ln(N) = \ln(N)$

Taking ln of both sides of $N = e^{\ln(N)}$

Gives: $\ln(N) = \ln\left(e^{\ln(N)}\right) = \ln(N)$, because ln and e^x undo each other.

⭐ Adventure 9 Story – The Age of Earth and the Planet Covered in Lead

Clair Patterson

Age of Earth

Thomas Midgley Jr. - Lead Gasoline, CFCs and Ferion

At the start of the 20th century, scientists still didn't know how old the Earth was. Estimates ranged wildly—from tens of millions to a billion years—but none were based on solid evidence. Clair Patterson, a quiet and extraordinarily careful geochemist, decided to answer this ancient question. His clock of choice was **uranium**, a radioactive element that decays into lead at a steady, predictable rate. By comparing the ratio of uranium to lead inside tiny zircon crystals, Patterson realized he could calculate the true age of Earth. But first he had to solve a shocking mystery.

Everywhere Patterson looked—air, water, dust, the surfaces of his instruments—he found **modern lead contamination**. Even the ancient zircon crystals contained far too much lead. Something was covering the entire planet in a thin film of this toxic metal, and it made his measurements impossible.

The trail led to **Thomas Midgley Jr.**, a brilliant but disastrous engineer. To stop car engines from knocking, Midgley invented tetra-ethyl lead gasoline. It worked perfectly, but it released lead into the air of every city and household in America. He later invented CFCs, chemicals that would eventually be found to damage the ozone layer.

To escape this contamination, Patterson built the **world's first ultra-clean laboratory**, the model for all future clean rooms. Inside this pristine space he finally obtained the measurements he needed. In 1956 he calculated the age of Earth at **4.55 billion years**, astonishingly close to today's accepted value. At the same time, he revealed the staggering amount of industrial lead entering the environment. Oil and chemical companies attacked his reputation, cut his funding, & fought his findings for decades, but He refused to back down. His persistence helped drive the creation of the **Clean Air Act**, which finally removed lead from gasoline. Within a generation, lead levels in children's blood fell by more than 90%.

Meanwhile, in a strange twist of poetic justice, Midgley was eventually killed by one of his own inventions—a rope-and-pulley device he designed after contracting polio. He became tangled in the machinery and died, a tragic end to a gifted but catastrophic inventor.

⭐ Ice Cores: Earth's Other Clock: While uranium gives the deep-time age of our planet, **ice cores** reveal more recent history. Each year snowfall freezes into a new layer, trapping bubbles of ancient air. Scientists read these layers like pages of a diary—volcanoes, climate swings, carbon dioxide, methane, even nuclear fallout. Beginning in the 1960s, Patterson examined these records and found a dramatic jump in atmospheric lead starting around **1927**—the exact year tetra-ethyl lead was introduced into gasoline. Patterson used these results extensively to fight the big oil companies and get the Lead out of Gasoline. After the Clean Air Act, ice cores showed lead levels dropping just as quickly.

Together, uranium dating and ice-core analysis show the sweep of Earth's story: from billions of years of radioactive decay to the rapid human changes of the last century. Patterson, who merely set out to measure Earth's age, ended up protecting the health of millions and reshaping modern environmental science.

⭐ Adventure 9 — Student Activity Sheets Solutions

Exponential Decay & the Age of the Earth - Using U-235, Clair Patterson, and the DiVA Triple Chart

Part 1. The derivative of e^x is e^x

Before we start the U-235 activity, let's look at the function that makes everything work:

$$e^x$$

Mathematicians discovered that e^x can be written as an infinite polynomial, called a Taylor series. Here are the first few terms:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Apply the power rule to show: $\frac{d(e^x)}{dx} = e^x$

$$\frac{d(e^x)}{dx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Then Apply the chain rule to see that: $\frac{d(e^{-x})}{dx} = -e^{-x}$

Part 2. Finding the Decay Constant from the Half-Life of U-235

We model radioactive decay with the exponential function:

$$M(t) = M_0 e^{-kt}$$

Where:

- $M(t)$ is the mass left after time t,
- M_0 is the starting mass,
- k is the decay constant,
- t is in billions of years.

U-235 has a half-life of: $T_{\frac{1}{2}} = 0.704$ **billion years.**

By definition: $M\left(T_{\frac{1}{2}}\right) = \frac{1}{2} M_0.$

Step 1: Plug the half-life into the decay formula

Start with: $M(t) = M_0 e^{-kt}.$

Now plug in: $t = T_{\frac{1}{2}} = 0.704$ ➔ $M_0 e^{-k(0.704)} = \frac{1}{2} M_0$

(Write $\frac{1}{2} M_0$ in the blank.)

Step 2: Cancel M_0 $$e^{-0.704k} = 1/2$$

(Write $\frac{1}{2}$in the blank.)

Step3. Take natural logs (ln): $$-0.704k = \ln\left(\frac{1}{2}\right) = -ln(2)$$

Solve for k: $$k = \frac{\ln(2)}{0.704}$$

Step 4. Calculate k **Use $\ln 2 \approx 0.693$:**

$$k = \frac{0.693}{0.704} \approx 0.984$$

(This should come out close to 0.9846.)

Step 5. Final simplified decay model: ***Here we replace k by 1 as it is very close to it?***

With $M_0 = 1024$ grams and $k \approx 1$, we use:

$$\boxed{M(t) = 1024\, e^{-t}}$$

Part 3 — Completing the Half-Life Table

The half-life of U-235 is: 0.704 billion years $= 704$ million years.

Fill in the amount left after each half-life.

Half-Life Table

Time Passed (billion yrs)	Half-Life n	U-235 Left (grams)
0.000	0	1,024
0.704	1	512
1.408	2	256
2.112	3	128
2.816	4	64
3.520	5	32
4.224	6	16
4.928	7	8

You will see the amount of Uranium-235 drops by ½ each half-life

When Does It Drop Below 11 grams?

Your table shows:

- After 6 half-lives: **16 g**
- After 7 half-lives: **8 g**

So **11 g** happens **between** the 6th and 7th half-life.

Find the halfway point:

$$\textbf{number of half} - \textbf{lie} \approx 6.5 \text{ half-lives.}$$

Compute the age:

$$6.5 \times 0.704 \approx 4.58 \textbf{ billion years}$$

This should be close to the age of the Earth: **4.6 billion years**.

Estimate the Age of the earth from the Chart

On the **green curve**, find the time when:

$$M(t) \approx 11 \textbf{ grams.}$$

Estimate:

$$t_{\text{from chart}} \approx 4.6 \textbf{ billion years.}$$

Is this close to your half-life estimate (from 6.5 × 0.704=4.58)? **Yes** **No**

Part 4 — Reading the Charts

Use your **DiVA triple chart** titled: **Exponential Radioactive Decay of U-235**

How a Tiny Spoonful of U-235 Helps Reveal Earth's Age

These three curves show how U-235 decays over billions of years.

Step 1. Lead Accumulated: Introducing L(t)

When uranium decays, the "missing" mass becomes lead.

We call this amount:

$$L(t) = M(0) - M(t).$$

The curve for L(t):

- starts at 0,
- Rises quickly at first,
- rises more slowly later,
- finally approaches 1024 grams.

Because lead grows exactly as uranium shrinks, the decay-speed curve gives:

$$\frac{dL}{dt} = v(t).$$

So: **v(t) is the slope of L(t),**

and the shaded area under v(t) tells how much lead has formed.

Step 2. Amount Left: $M(t) = M(0) - L(t)$ (green dashed curve)

- Starts at **1024 grams** - Shrinks as time passes on (increases)- Shows how much U-235 is still in the rock. For Lead the opposite will happen starting at 0 and ending at 1024

What is the mass at:

- **$t = 0$ billion years?** $\qquad M(0) = 1024$ **g** $\quad$ **L(0) =0 g**
- **$t = 1$ billion years?** $\qquad M(1) \approx 380$ **g** $\quad$ **L(1)** ≈ 644 **g**
- **$t = 2$ billion years?** $\qquad M(2) \approx 140$ **g** $\quad$ **L(2)** ≈ 884 **g**

(Use the green curves. Exact numbers are not required — estimates are encouraged.)

Step 3. Decay Speed: $v(t)$(red curve)

$$v(t) = \text{decay speed} = \frac{d(L(t))}{dt} == \frac{d(1024 - 1024e^{-t})}{dt} = \frac{d(-1024e^{-t})}{dx} = 1024e^{-t}$$

V(t) is the speed of disappearance of U-235 or how many grams Lead appear each billion years.
From the red curve, estimate the decay speed at:

- **$v(0)$: 1024 g/billion yr**
- **$v(1)$: 380 g/billion yr**
- **$v(2)$: 140 g/billion yr**

Part 5 — Checking the Mass at $t = 1$Using an Integral

The decay-speed curve

$$v(t) = 1024e^{-t}$$

tells how fast uranium is disappearing.

So the area under v(t) from 0 to 1 gives the total mass of U-235 lost:

$$L(1) = \int_0^1 v(t)\, dt.$$

Compute the loss:

$$L(1) = \int_0^1 1024\, e^{-t} dt \approx 647 \text{ grams.}$$

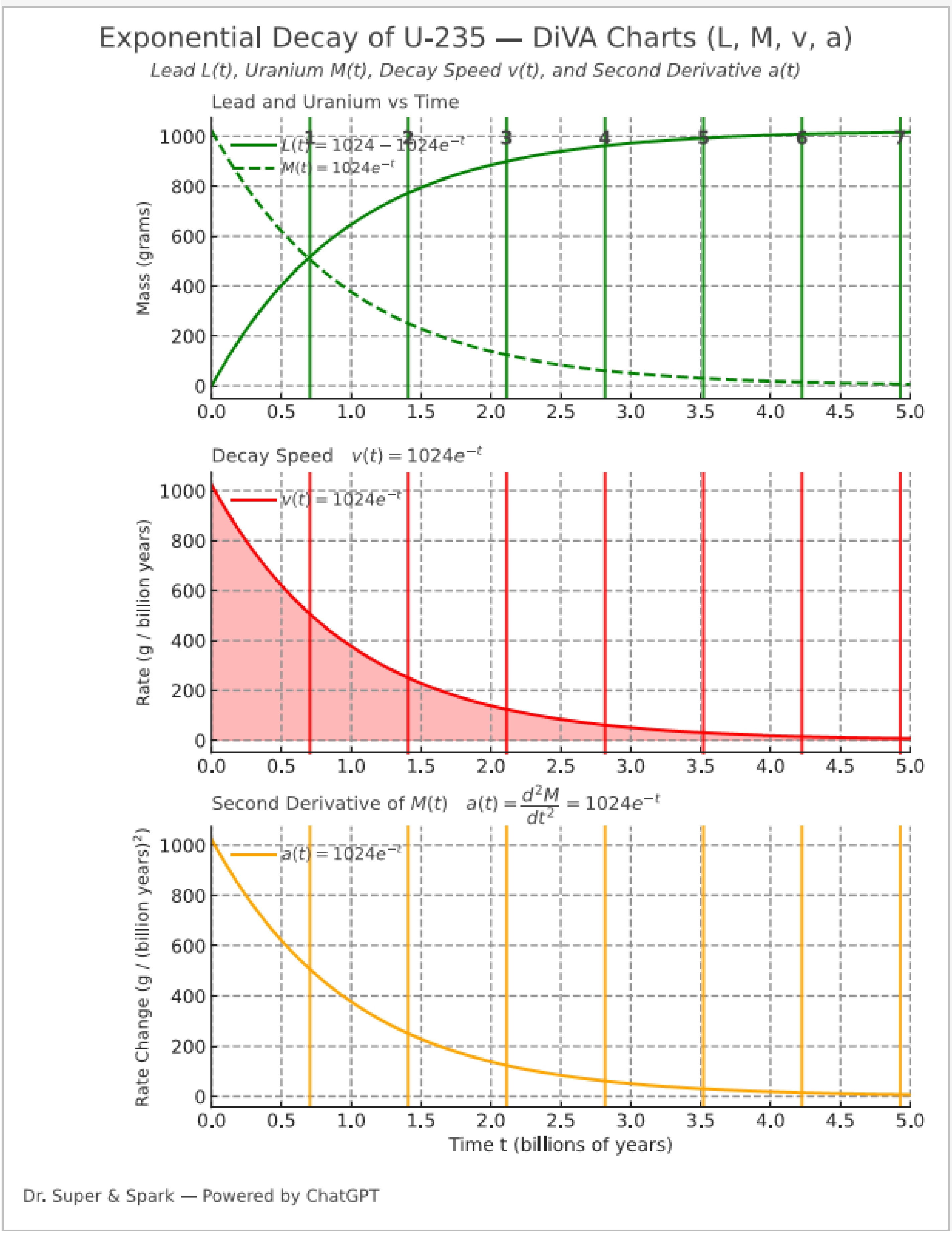
Convert this into uranium remaining – Use:

$$\mathrm{M(1) = M(0) - L(1)},$$

$$M(1) = 1024 - 647 = 377 \text{ grams.}$$

Check the chart

The curve for M(t) at $t = 1$ is close to 377 grams, so the integral and the graph agree.


Exponential Decay of U-235 — DiVA Charts (L, M, v, a)
Lead L(t), Uranium M(t), Decay Speed v(t), and Second Derivative a(t)
Lead and Uranium vs Time
$L(t) = 1024 - 1024e^{-t}$
$M(t) = 1024e^{-t}$
Mass (grams)
Decay Speed $v(t) = 1024e^{-t}$
$v(t) = 1024e^{-t}$
Rate (g / billion years)
Second Derivative of $M(t)$ $a(t) = \frac{d^2M}{dt^2} = 1024e^{-t}$
$a(t) = 1024e^{-t}$
Rate Change (g / (billion years)^2)
Time t (billions of years)
Dr. Super & Spark — Powered by ChatGPT

⭐ Adventure 9– Student Activity Sheets

Exponential Decay & the Age of the Earth - Using U-235, Clair Patterson, and the DiVA Triple Chart

Part 1. The derivative of e^x is e^x

Before we start the U-235 activity, let's look at the function that makes everything work:

$$e^x$$

Mathematicians discovered that e^x can be written as an infinite polynomial, called a Taylor series. Here are the first few terms:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Apply the power rule to show: $\frac{d(e^x)}{dx} = e^x$

$\frac{d(e^x)}{dx} =$ ______________________________

Then Apply the chain rule and find: $\frac{d(e^{-x})}{dx} =$ ______________

Part 2. Finding the Decay Constant from the Half-Life of U-235

We model radioactive decay with the exponential function here the independent variable is t instead of x:

$$M(t) = M_0 e^{-kt}$$

Where:

- $M(t)$ is the mass left after time t,
- M_0 is the starting mass,
- k is the decay constant,
- t is in billions of years.

U-235 has a half-life of: $T_{\frac{1}{2}} = 0.704$ **billion years.**

By definition: $M\left(T_{\frac{1}{2}}\right) = \frac{1}{2} M_0.$

Step 1: Plug the half-life into the decay formula

Start with: $M(t) = M_0 e^{-kt}.$

Now plug in: $t = T_{\frac{1}{2}} = 0.704$ ➔ $M_0 e^{-k(0.704)} =$ ______________

Step 2: Cancel M_0

$$e^{-0.704k} = \text{______________}$$

Step3. Take natural logs (ln) on both sides of the equation:_ ______________ = ______________

Solve for k: $k =$ __________

Step 4. Calculate k Use $\ln 2 \approx 0.693$:

$k \approx$ $(3 \ decimals)$

Step 5. Final simplified decay model: ***Is k is close to 1?*** ***Yes*** ***No***

With $M_0 = 1024$ grams and $k \approx 1$ rewrite M(t)

$$M(t) = M_0 e^{-kt} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Part 3 — Completing the Half-Life Table

The half-life of U-235 is: 0.704 billion years $= 704$ million years.

Fill in the amount left after each half-life in the following Half-Life Table below

Time Passed (billion yrs)	Half-Life n	U-235 Left (grams)
0.000	0	1,024
0.704	1	
1.408	2	
2.112	3	
2.816	4	
3.520	5	
4.224	6	
4.928	7	

(What pattern do you see?____________________

When Does It Drop Below 11 grams?

When **Clair Patterson** did his experiment he found 11 grams of U-235 were left out of 1024 grams.

This corresponds to approximately how many half-lives?

Your table shows:

- After 6 half-lives U-235 is: ________ g
- After 7 half-lives U-235 is: ________ g

$$\text{Number of half} - \text{lives} \approx \_\_\_\_\_\_\_\_\_\_\_\_$$

Compute the age of Earth by multiplying this number by 0.704 billion years:

Age of Earth = __________ x 0.704 = __________ billion years

Estimate the Age of the earth from the Chart

On the **green curve**, find the time when:

$$M(t) \approx 11 \text{ grams.}$$

Estimate:

$$t_{\text{from chart}} \approx \_\_\_\_\_ \text{ billion years.}$$

Is this close to your half-life estimate that you found? **Yes** **No**

🧮 Part 4 — Reading the Charts

Use your **DiVA triple chart** titled: **Exponential Radioactive Decay of U-235**

How a Tiny Spoonful of U-235 Helps Reveal Earth's Age

These three curves show how U-235 decays over billions of years.

Step 1. Lead Accumulated: Introducing L(t)

When uranium decays, the "missing" mass becomes lead.

We call this amount:

$$\boldsymbol{L(t) = M(0) - M(t).}$$

The curve for L(t):

- starts at 0, Rises quickly at first, rises more slowly later, finally approaches 1024 grams.
- Because lead grows exactly as uranium shrinks, the decay-speed curve gives:

$$\boldsymbol{\frac{dL}{dt} = v(t).}$$

So: **v(t) is the slope of L(t),**

and the shaded area under v(t) tells how much lead has formed.

🧑‍🏫 Adventure 10– L'Hôpital's Rule, Ratios, & Feynman–Gamow Space Mission

🧩 Purpose

Adventure 9 introduces students to:

- Buffon's Needle Problem (Using integrals to find probabilities)
- Ratios that produce $0/0$and ∞/∞forms
- The idea of comparing *rates* rather than *values*
- L'Hôpital's Rule in a conceptual, derivative-free explanation ("which part changes faster?")
- The DiVA structure (Distance → Velocity → Acceleration) as a natural realization of L'Hôpital's Rule
- Long-time behavior of functions and ratios

Students explore these ideas through the fictional but physically-grounded **Feynman–Gamow probe mission**, using distance–velocity–acceleration charts and tables.

🧱 Prerequisites

Students should know:

- Derivative meaning (rate of change), from Sections 6 & 7
- How to read the DiVA triple chart
- Basic polynomial functions and tables
- Basic experience marking points on coordinate axes

No formal limit theory is required.

📚 Set-up

Material for Students:

- The Feynman–Gamow story sheet
- The Feynman–Gamow **Student DiVA Chart**
- Buffon's Needle Problem Activity Sheet
- Adventure 10 Student Activity Sheets
- Ruler and pencil
- Optional: calculators for long-time ratio tables

Material for Teachers:

- Adventure 10 Student Activity Sheets Solution
- The **Teacher DiVA Chart** showing all curves and ratio points clearly labeled

Begin by having students read or listen to the story aloud. The mission framing helps anchor the mathematics in real "stakes."

🧮 Guided Procedure

1. Warm-Up: The NASA Rule

Students learn that:

$$\frac{f(t)}{g(t)} > 2 \text{for all } t$$

must remain true during the mission to avoid signal interference.

They see from the story why a ratio that approaches 2 but never dips below it is physically meaningful.

2. Computing Velocities and Accelerations

Students compute:

$$f(t) = 2t^2 + 175t, g(t) = t^2 + 5t$$

Velocity:

$$f'(t) = 4t + 175, g'(t) = 2t + 5$$

Acceleration:

$$f''(t) = 4, g''(t) = 2$$

Teaching Point:
Acceleration ratio is already the limit.
This is the key visual realization of L'Hôpital's Rule.

3. Reading the Distance and Velocity Charts

Using their five marked points at $t = 5,10,20,30,50$, students:

- Draw f(t) and g(t)
- Interpret the curvature (increasing slope)
- Observe the distance ratio falling toward ~2

Teaching Point: The distance functions do not reveal the long-term ratio directly — their early behavior is misleading, and the ratio changes dramatically.

4. Ratio Tables and L'Hôpital Reasoning

Students fill in:

- Distance ratios (given)
- Velocity ratios (computed)
- Acceleration ratio (constant = 2)

Teaching Points:

- The acceleration ratio **controls** the long-time behavior.
- This matches the conceptual version of L'Hôpital's Rule:
  "Compare how fast each part is changing."

5. Long-Time Tables (t = 240, 720, 1200)

Students compute values showing the distance and velocity ratios flattening toward 2.

Teaching Point:

Even though f(t) and g(t) are huge numbers, their **ratio becomes simple**.

6. The Chain Reaction Explanation

This is the heart of Section 9.

Acceleration ratio → Velocity ratio → Distance ratio

This mirrors L'Hôpital's Rule:

$$\lim_{t\to\infty}\frac{f(t)}{g(t)}=\lim_{t\to\infty}\frac{f'(t)}{g'(t)}=\lim_{t\to\infty}\frac{f''(t)}{g''(t)}=2$$

Even though we never state the formal rule to younger students, the DiVA chart **is** L'Hôpital's Rule visually.

🪄 Discussion & Reflection

Use these questions:

- Why does acceleration completely determine the long-time ratio?
- Why don't early distance ratios tell you the whole story?
- Why does dividing huge numbers sometimes hide simple truths?
- How is this similar to comparing rockets, cars, or runners with different long-term speeds?

This helps students internalize the "rate-of-change" approach.

🧑‍🏫 Teacher Tips

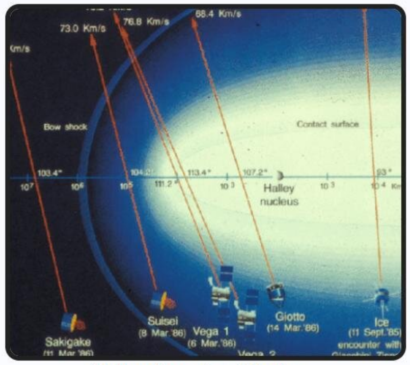
- Emphasize the *visual* structure: distance curves curve upward, velocity lines tilt upward, acceleration is constant.
- Encourage students to notice pattern: every derivative step simplifies the situation.
- When students compute the ratios, always link back to the DiVA charts.
- For stronger students: mention that L'Hôpital's Rule is the *formal* version of what they just discovered.

⭐ Optional Extension

Use the **Newton–Leibniz cubic challenge** included in the student document:
It gives a hands-on derivative table where the 3rd derivative reveals the limiting ratio (3).

🚀 Adventure 10 Story — The Feynman–Gamow Deep-Space Mission


Richard Feynman



Halley Armada - 5 Probes


George Gamow

Two Legendary Physicists, One Comet, and a Race Against Interference

In 2061, Halley's Comet sweeps past Earth for the first time since 1986.

NASA prepares a daring mission: launch two coordinated deep-space probes to intercept the comet and study it up close as it races through the inner solar system. To honor two of the greatest physicists of the 20th century, the probes are named after:

Richard Feynman (1918–1988) A Nobel Prize–winning physicist known for his brilliant imagination, playful personality, and ability to explain deep ideas with simple pictures. Feynman transformed quantum mechanics, helped uncover the cause of the Challenger disaster, and inspired generations with his joy for discovery.

The Feynman Probe carries a *quantum telescope*, the most sensitive instrument ever flown near a comet. It can detect microscopic dust and plasma streams trailing behind Halley's nucleus.

George Gamow (1904–1968) A physicist with enormous creativity who helped explain radioactive decay, pioneered early ideas about black holes, and first predicted the *cosmic microwave background*—the leftover glow of the Big Bang. He also wrote imaginative science books that made hard ideas fun.

The Gamow Probe carries the mission's *communication and navigation systems*, sending powerful signals back to Earth to steer both spacecraft during their high-speed encounter with the comet.

The Problem NASA Discovered

Because the probes travel along similar paths, NASA engineers found a serious issue:

If the Feynman Probe ever becomes less than twice as far from Earth as the Gamow Probe, the radio signals from Gamow can interfere with Feynman's delicate quantum measurements.

To protect the telescope, the mission must obey a strict separation rule: **The Feynman–Gamow distance ratio must approach 2 but never go below 2.** $\frac{f(t)}{g(t)} \geq 2$

So engineers design the propulsion systems carefully:

The **Feynman Probe** receives a long-range ion engine with **more sustained acceleration** than the Gamow Probe. The **Gamow Probe** accelerates more gently but communicates more strongly.

Both probes launch together on **July 28, 2061**, beginning their chase toward Halley's blazing tail. But even with careful engine design, Mission Control needs to know: Could Gamow's signals drift too close and contaminate Feynman's telescope?

To see if the ratio truly approaches 2 and stays above 2 for the entire mission, you will analyze their motion using **DiVA**: **Distance → Velocity → Acceleration** charts using L'Hôpital's Rule that should reveal whether the mission is safe or not?

🚀 Adventure 10 — Student Worksheet Solutions

The Feynman–Gamow Mission: Ratios, Change, and DiVA

🚀 Mission Story

The year is 2061, and Halley's Comet is returning to Earth. NASA launches two research probes:

🔶 Feynman Probe

Fast, powerful, designed to collect high-energy dust.

🔷 Gamow Probe

Gentle, designed to map the comet's gas cloud.

🚀 PART 1 — Understanding the NASA Functions and Requirements

Functions and Requirements

The distances (in kilometers) of the probes from Earth are:

$$\boldsymbol{f(t) = 2t^2 + 175t} \text{ (Feynman)}$$

$$\boldsymbol{g(t) = t^2 + 5t} \text{ (Gamow)}$$

where $\boldsymbol{t}$ is time in hours after launch.

To keep their scientific instruments from interfering, NASA sets a strict rule:

$$\boldsymbol{\frac{f(t)}{g(t)} \, must\ become\ very\ close\ to\ 2\ but\ eventually\ \frac{f(t)}{g(t)} \geq 2} \text{ for all time.}$$

Your job is to use distance, velocity, acceleration, and ratios to discover *why this is always true*.

Before we study ratios, we must understand how distance changes.

2. Derivatives: Velocity and Acceleration

The velocity of a probe tells how fast its distance is changing at each moment.

The acceleration tells how fast the velocity is changing.

We write velocity using a new notation:

$$\text{Velocity of } \boldsymbol{f} = \boldsymbol{\frac{df(t)}{dt} = f'(t)}$$

$$\text{Velocity of } \boldsymbol{g} = \boldsymbol{\frac{dg(t)}{dt} = g'(t)}$$

We write acceleration as the second derivative:

$$\boldsymbol{f''(t), g''(t)}$$

Your Task: Calculate the Derivatives

Use the formulas for $f(t) = 2t^2 + 175t$ and $g(t) = t^2 + 5t$ to compute:

$f'(t) =$ 4t+175

$g'(t) =$ 2t+5

$$f''(t) = 4$$

$$g''(t) = 2$$

(You will use these later in the ratio charts.)

⭐ PART 2 — Distance Table

Use the times already marked on your chart:

t (hours)	Feynman $f(t)$	Gamow $g(t)$	Ratio $f(t)/g(t)$
5	925	50	18.50
10	1,950	150	13.00
20	4,300	500	8.60
30	7,050	1050	6.71
50	13,750	2750	5.00

Your Task: Draw a smooth curve through the five red ratio points on your Distance Ratio Chart.

📐 PART 3 — Velocity Table (Students Compute)

Use your derivative formulas:

$$f'(t) = \frac{df(t)}{dt} = 4t + 175 \qquad g'(t) = \frac{dg(t)}{dt} = 2t + 5$$

Compute the velocities and the ratios (with 2 decimal):

t (hours)	Feynman $f'(t)$	Gamow $g'(t)$	Ratio $f'(t)/g'(t)$
5	195	15	13.00
10	215	25	8.60
20	255	45	5.67
30	295	65	4.54

t (hours)	Feynman $f'(t)$	Gamow $g'(t)$	Ratio $f'(t)/g'(t)$
50	375	105	3.57

Your Task: Plot your five points for f'(t) and g'(t) each and mark their values on the Velocity and Velocity Ratio Chart and draw a line for each function f'(t) and g'(t) and find the slope of these two lines.

Slope of f'(t) = $\frac{215-195}{5} = 4$ Slope of g'(t) = $\frac{25-15}{5} = 2$

📈 PART 4 — Acceleration Table

Your second derivatives:

$$f''(t) = 4, g''(t) = 2$$

Fill in the table:

t (hours)	$f''(t)$	$g''(t)$	Ratio $f''(t)/g''(t)$
5	4	2	2
10	4	2	2
20	4	2	2
30	4	2	2
50	4	2	2

Your Task: Plot the five identical points for each probe and draw the constant acceleration lines.

🔭 PART 5 — Long-Time Behavior

We now look far into the future of the mission as the probes follow the comet:

Distance Values Only (Compute the ratios with 2 decimals)

t (hours)	Feynman $f(t)$	Gamow $g(t)$	Ratio $f(t)/g(t)$
240	157,200	58,800	2.67
720	1,162,800	522,000	2.23
1200	3,090,000	1,446,000	2.14

Question 1: As time becomes very large, what number do your new ratios seem to approach?

The ratio is approaching: 2

Velocity Values Only

t (hours)	$f'(t) =$ 4t+175	$g'(t) =$ 2t+5	Ratio $f'(t)/g'(t)$
240	1,135	485	2.34
720	3,055	1445	2.11
1200	4,975	2405	2.07

Question 2: Do the velocity ratios seem to approach the same number as the long-time distance ratios?

Yes No

Acceleration Ratio – Compute:

$$\frac{f''(t)}{g''(t)} = 2$$

Question 3a: This ratio stays the same forever. What does that tell you about where the velocity ratio is heading as time grows? Velocity Ratio is heading to: 2 also

Question 3b: If the velocity ratio is heading toward the same number as the acceleration ratio, what does that imply about the distance ratio? Distance Ratio is also approaching: 2

Question 3c: Explain the chain reaction: Constant Acceleration Ratio → velocity ratio → distance ratio.

As t gets large velocity ratio will also approach 2 as the velocities are areas under the lines f''(x)=4 and g''(x)=2 the ratio is initially different as f' has 175 and g' only a 5 constant added to it. For the same reason since the velocity ratio approaches 2 that will force the ratio of the distance to approach 2 also.

📘 PART 6 — Final Mission Explanation

Write 3–4 short sentences:

What happens to the distances of the two probes?

What happens to the distance ratio over time?

Why does the ratio never go below 2?

How did the L'Hôpital's Rule and DiVA charts help you understand this?

The distances of both probes increase without bound, but the ratio approaches 2 from above.Since the acceleration ratio is always 2, the velocity ratio must approach 2, and therefore the distance ratio must also approach 2.

⭐ **Challenge Question — L'Hôpital's Rule in Action!**

Two new probes, Newton and Leibniz, are launched on a test flight.

Their distances from Earth are modeled by:

$$f(t) = 3t^3 + 2t^2 - 5t$$

$$g(t) = t^3 - 4t + 1$$

Both distances grow very large as t becomes large — so the question becomes:

🔍 **What number does the ratio $\frac{f(t)}{g(t)}$ get closer and closer to as $t \to \infty$?**

💡 Use the L'Hôpital's Rule Derivative Table

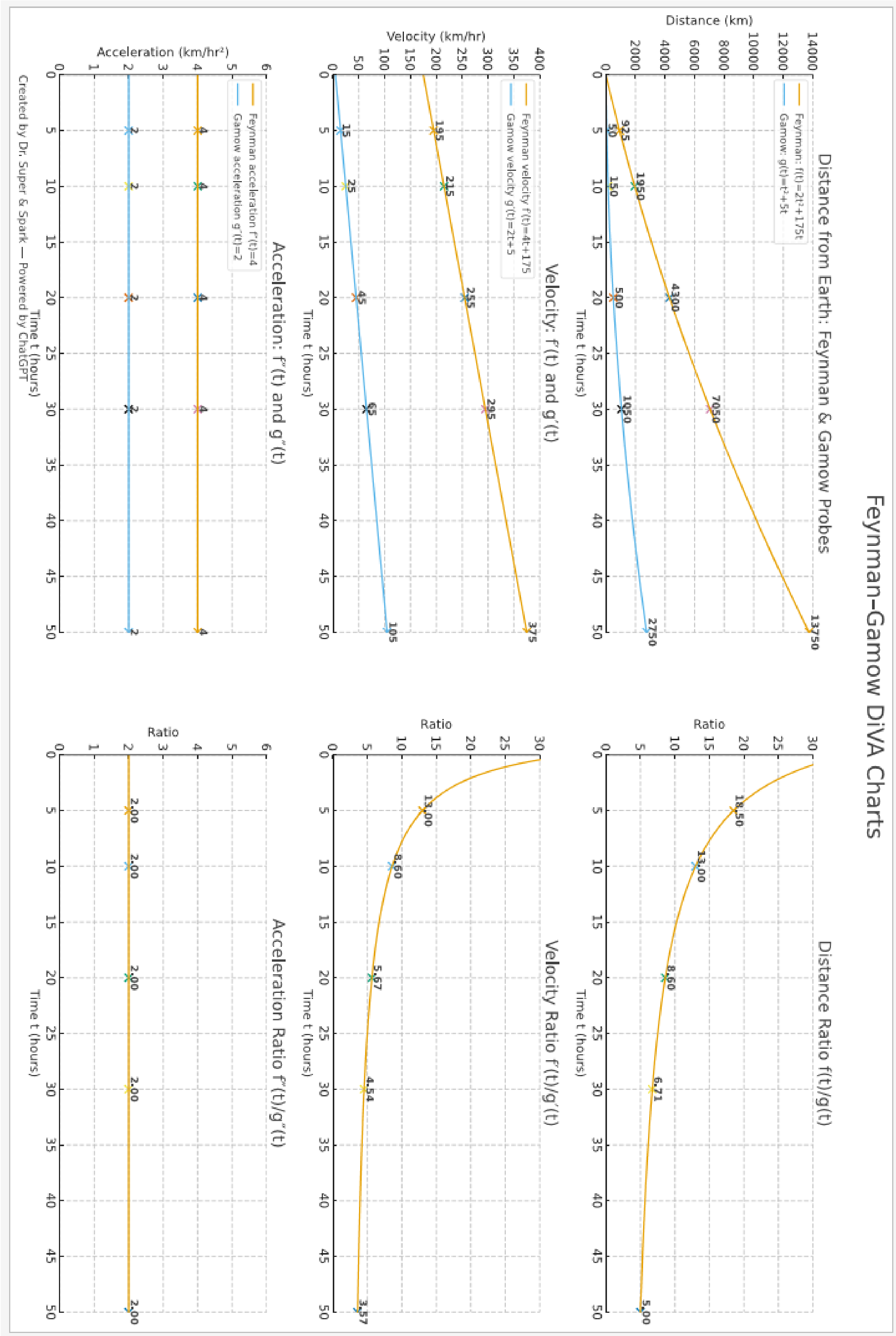
Order of Derivative	**Newton**	**Leibniz**	Ratio as **t** → ∞
original function	$f(t) = 3t^3 + 2t^2 - 5t$	$g(t) = t^3 - 4t + 1$	$\lim_{t\to\infty} \frac{f(t)}{g(t)} = \frac{\infty}{\infty}$
1st derivative	$f'(t) = 9t^2 + 4t$	$g'(t) = 3t^2 - 4$	$\lim_{t\to\infty} \frac{f'(t)}{g'(t)} = \frac{\infty}{\infty}$
2nd derivative	$f''(t) = 18t$	$g''(t) = 6t$	$\lim_{t\to\infty} \frac{f''(t)}{g''(t)} = \frac{\infty}{\infty}$
3rd derivative	$f'''(t) = 18$	$g'''(t) = 6$	$\lim_{t\to\infty} \frac{f'''(t)}{g'''(t)} = 3$

💡 So what is the answer? L'Hôpital's Rule says it is $\lim_{t\to\infty} \frac{f'''(t)}{g'''(t)} = 3$

💡 In general if you are dividing two polynomials of the same power the ratio will approach the ratio of the coefficients of the highest powers.

$$\lim_{t\to\infty} \frac{10t^5 + 2t^2 - 5t}{2t^5 - 4t} = \frac{10}{2} = 5$$

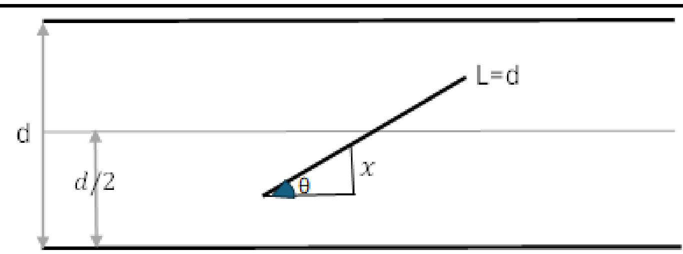
🚀 Adventure 10— Student Chart Solution



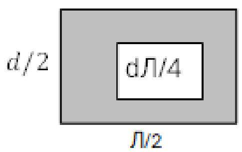
🚀 Adventure 10— Buffon's Needle Problem Solution

Buffon's Needle Problem Solution

What is the probability that the needle will cross a line?

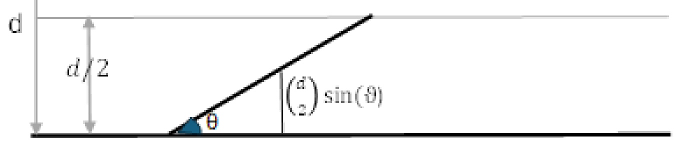


1. we will look at 0≤x≤d/2
2. We will then consider 0≤θ≤Л/2

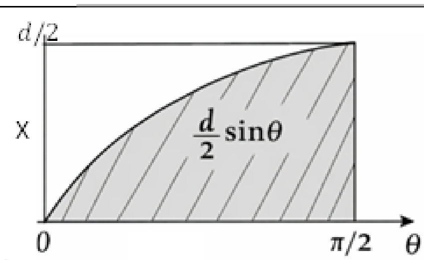


All possible positions for the needle given the conditions (1) and (2)

$$\int_0^{d/2} dx = d/2 \Longrightarrow \int_0^{Л/2} (d/2)d\vartheta = dЛ/4$$



All possible positions for the needle to cross the bottom line.



$$\int_0^{\frac{d}{2}\sin(\theta)} dx = \frac{d}{2}\sin(\theta)$$

$$Crossing\ drops = \int_0^{\frac{\pi}{2}} \frac{d}{2}\sin(\theta)d\,\boldsymbol{\theta} = d/2 \int_0^{\frac{\pi}{2}} \sin(\theta)d\,\boldsymbol{\theta} = \textbf{d/2}$$

Prob Crossing =(d/2)/(dЛ/4)=2d/dЛ=2/Л~=2/3

In General L < d Prob Crossing =(L/2)/(dЛ/4)=**2L/dЛ**

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🚀 Adventure 10— Buffon's Needle Problem Solution

Goal: Estimate π using geometry + probability.

Materials:

Paper (with parallel lines), ruler (or printed sheet), and a "needle" (toothpick)

OR use an online simulation / random generator (optional)

Setup

Parallel lines are spaced d units apart. A needle of length L is dropped at random.

For this activity we use the special case: L = d

Part A — Define the random variables

Define:

x = distance from the needle's center to the nearest line

θ = angle the needle makes with the lines

Give the ranges:

$0 \le x \le d/2$

$0 \le \theta \le \pi/2$

Part B — All possible positions for the needle given (1) and (2)

Compute the following two integrals and show that all the possible positions for the needle given (1) and (2) is d π/4 their product. You can also see that from the rectangle that has sides d/2 and π/4

First Integrate from 0 to d/2

$$\int_0^{d/2} dx = d/2$$

Integrate the result from $\mathbf{0}$ ***to*** $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} (d/2)d\theta =$$

(d/2)(π/2) = d π/4

Part C— When does the needle cross a line given (1) and (2)?

Use the picture and explain why the needle crosses the nearest line when L = d:

$$x \le \frac{d}{2}\sin(\theta)$$

In the (θ, x)-plane, explain that "Crossing drops" are the region under the curve $x = \frac{d}{2}\sin(\theta)$.

First integrate from 0 to $\frac{d}{2}\sin(\theta)$:

$$\int_0^{\frac{d}{2}\sin(\theta)} dx = \frac{d}{2}\sin(\theta)$$

$$\textit{Then integrate the result from } 0 \textit{ to } \frac{\pi}{2}$$

$\textit{Crossing drops} = \int_0^{\frac{\pi}{2}} \frac{d}{2}\sin(\theta) d\,\theta = \left(\frac{d}{2}\right)\left[-\cos(0) - (-\cos\left(\frac{\pi}{2}\right))\right] = \left(\frac{d}{2}\right)[0-(-1)] = \left(\frac{d}{2}\right)[1]$ =d/2

Note: In general, if d is not equal to L the crossing condition becomes:

$$x \leq \frac{L}{2}\sin(\theta)$$

So the Cross drops will give L/2

Part D — The probability result

Use the area ratio argument to show:

$$P(\text{cross}) = \frac{2}{\pi} \text{ (when } L = d) = 0.637$$

P(cross)= 2L/dЛ (When L < d)

Explain why an experiment with N drops and C crossings gives:

$$\hat{P} = \frac{C}{N}$$

So the estimate for π is:

$$\pi \approx \frac{2}{\hat{P}} = \frac{2N}{C}$$

Part D — Do an experiment (or simulation)

Perform an experiment (throw the needle at least 20 times). Record:

N =_20, 50

$$C = 12, 32$$

$$\pi \approx \frac{2N}{C} = 0.333, 3.125$$

Repeat with a larger N. Did your estimate get closer to π? Yes it did!

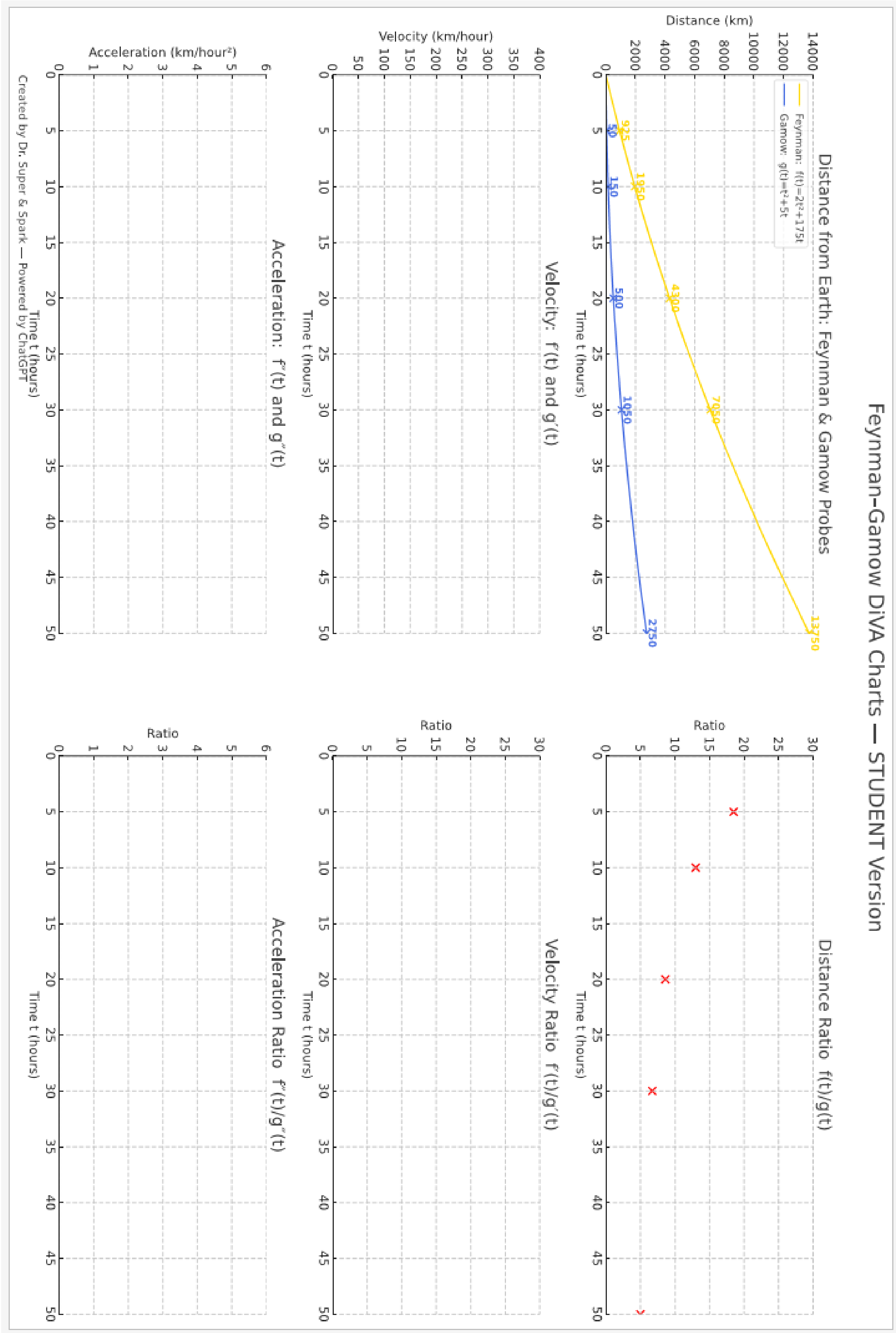
Extension (optional)

If $L < d$, the general formula is: $P(\text{cross}) = \frac{2L}{\pi d}$

Challenge: If you double the needle length L, what happens to the probability? Explain in one sentence.

As long as double the length is less than D the probability $P(\text{cross}) = \frac{2L'}{\pi d}$ doubles the original if L'>d then

Adventure 10— Student Chart



🚀 Adventure 10— Student Worksheet

The Feynman–Gamow Mission: Ratios, Change, and DiVA

🚀 Mission Story

The year is 2061, and Halley's Comet is returning to Earth. NASA launches two research probes:

🔶 Feynman Probe

Fast, powerful, designed to collect high-energy dust.

🔷 Gamow Probe

Gentle, designed to map the comet's gas cloud.

🚀 PART 1 — Understanding the NASA Functions and Requirements

Functions and Requirements

The distances (in kilometers) of the probes from Earth are:

$$f(t) = 2t^2 + 175t \text{ (Feynman)}$$

$$g(t) = t^2 + 5t \text{ (Gamow)}$$

where t is time in hours after launch.

To keep their scientific instruments from interfering, NASA sets a strict rule:

$$\frac{f(t)}{g(t)} \textit{ must become very close to 2 but eventually } \frac{f(t)}{g(t)} \geq 2 \text{ for all time.}$$

Your job is to use distance, velocity, acceleration, and ratios to discover *why this is always true*.

Before we study ratios, we must understand how distance changes.

2. Derivatives: Velocity and Acceleration

The velocity of a probe tells how fast its distance is changing at each moment.

The acceleration tells how fast the velocity is changing.

We write velocity using a new notation:

$$\text{Velocity of } f = \frac{df(t)}{dt} = f'(t)$$

$$\text{Velocity of } g = \frac{dg(t)}{dt} = g'(t)$$

We write acceleration as the second derivative:

$$f''(t), g''(t)$$

Your Task: Calculate the Derivatives

Use the formulas for $f(t) = 2t^2 + 175t$ and $g(t) = t^2 + 5t$ to compute:

$f'(t) =$ ______________________________

$g'(t) =$ ______________________________

$f''(t) =$ ______________________________

$g''(t) =$ ______________________________

(You will use these later in the ratio charts.)

⭐ PART 2 — Distance Table

Use the times already marked on your chart:

t (hours)	Feynman $f(t)$	Gamow $g(t)$	Ratio $f(t)/g(t)$
5	925	50	18.50
10	1,950	150	13.00
20	4,300	500	8.60
30	7,050	1050	6.71
50	13,750	2750	5.00

Your Task: Draw a smooth curve through the five red ratio points on your Distance Ratio Chart.

📐 PART 3 — Velocity Table (Students Compute)

Use your derivative formulas:

$$f'(t) = df(t)/dt = \_\_\_\_\_\_\_\_\_\_\_\_ \quad g'(t) = dg(t)/dt = \_\_\_\_\_\_\_\_\_\_\_\_$$

Compute the velocities and the ratios (with 1 decimal):

t (hours)	Feynman $f'(t)$	Gamow $g'(t)$	Ratio $f'(t)/g'(t)$
5			
10			
20			
30			
50			

Your Task: Plot your five points for f'(t) and g'(t) each and mark their values on the Velocity and Velocity Ratio Chart and draw a line for each function f'(t) and g'(t) and find the slope of these two lines.

Slope of f'(t) = ___________ Slope of g'(t) = ___________

📈 PART 4 — Acceleration Table

Your second derivatives:

$$f''(t) = \_\_\_, g''(t) = \_\_\_$$

t (hours)	$f''(t)$	$g''(t)$	Ratio $f''(t)/g''(t)$
5			
10			
20			
30			
50			

Fill in the table:

Your Task: Plot the five identical points for each probe and draw the constant acceleration lines.

🔭 PART 5 — Long-Time Behavior

We now look far into the future of the mission as the probes follow the comet:

Distance Values Only (Compute the ratios with 2 decimals)

t (hours)	Feynman $f(t)$	Gamow $g(t)$	Ratio $f(t)/g(t)$
240	157,200	58,800	
720	1,162,800	522,000	
1200	3,090,000	1,446,000	

Question 1: As time becomes very large, what number do your new ratios seem to approach?

The ratio is approaching: ☐

Velocity Values Only

t (hours)	$f'(t) =$ 4t+175	$g'(t) =$ 2t+5	Ratio $f'(t)/g'(t)$
240			
720			
1200			

Question 2: Do the velocity ratios seem to approach the same number as the long-time distance ratios?

Yes No

Acceleration Ratio

Compute:

$$\frac{f''(t)}{g''(t)} = \_\_\_\_\_$$

Question 3a:

This ratio stays the same forever. What does that tell you about where the velocity ratio is heading as time grows? Velocity Ratio is heading to: ____________________

Question 3b:

If the velocity ratio is heading toward the same number as the acceleration ratio, what does that imply about the distance ratio? Distance Ratio is also approaching: _______________

Question 3c:

Explain the chain reaction:

Constant acceleration ratio → velocity ratio → distance ratio.

____

📘 PART 6 — Final Mission Explanation

Write 3–4 short sentences:

What happens to the distances of the two probes?

What happens to the ratios over time?

Why does the ratio never go below 2?

How did the L'Hôpital's Rule and DiVA charts help you understand this?

____

____

____

⭐ Challenge Question — L'Hôpital's Rule in Action!

Two new probes, Newton and Leibniz, are launched on a test flight.

Their distances from Earth are modeled by:

$$f(t) = 3t^3 + 2t^2 - 5t$$

$$g(t) = t^3 - 4t + 1$$

Both distances grow very large as t becomes large — so the question becomes:

🔍 What number does the ratio $\frac{f(t)}{g(t)}$ get closer and closer to as $t \to \infty$?

💡 Use the L'Hôpital's Rule Derivative Table

Order of Derivative	Newton	Leibnitz	Ratio as t → ∞
original function	$f(t) = 3t^3 + 2t^2 - 5t$	$g(t) = t^3 - 4t + 1$	$\lim_{t\to\infty} \frac{f(t)}{\text{g}(\text{t})} = \frac{\infty}{\infty}$
1st derivative	$f'(t) =$	$g'(t) =$	$\lim_{t\to\infty} \frac{f'(t)}{\text{g}'(\text{t})} =$
2nd derivative	$f''(t) =$	$g''(t) =$	$\lim_{t\to\infty} \frac{f''(t)}{\text{g}''(\text{t})} =$
3rd derivative	$f'''(t) =$	$g'''(t) =$	$\lim_{t\to\infty} \frac{f'''(t)}{\text{g}'''(\text{t})} =$

💡 So what is the answer? L'Hôpital's Rule says it is $\lim_{t\to\infty} \frac{f'''(t)}{\text{g}'''(\text{t})} = \square$

💡 In general if you are dividing two polynomials of the same power the ratio will approach the ratio of the coefficients of the highest powers.

$$\lim_{t\to\infty} \frac{10t^5 + 2t^2 - 5t}{2t^5 - 4t} = \frac{10}{2} = 5$$

🚀 Adventure 10 — Buffon's Needle Problem —Student Activity Sheet

Goal: Estimate π using geometry + probability.

Materials:

Paper (with parallel lines), ruler (or printed sheet), and a "needle" (toothpick)

OR use an online simulation / random generator (optional)

Setup

Parallel lines are spaced d units apart. A needle of length L is dropped at random.

For this activity we use the special case: L = d

Part A — Define the random variables

Define:

x = distance from the needle's center to the nearest line

θ = angle the needle makes with the lines

Give the ranges:

$0 \le x \le d/2$

$0 \le \theta \le \pi/2$

Part B — All possible positions for the needle given (1) and (2)

Compute the following two integrals and show that all the possible positions for the needle given (1) and (2) is d π/4 their product. You can also see that from the rectangle that has sides d/2 and π/4

First Integrate from 0 to d/2

$$\int_0^{d/2} dx =$$

(d/2)(π/2) = d π/4

$$\textit{Integrate the result from } 0 \textit{ to } \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} d\theta =$$

Part C— When does the needle cross a line given (1) and (2)?

Use the picture and explain why the needle crosses the nearest line when L = d:

$$x \le \frac{d}{2}\sin(\theta)$$

In the (θ, x)-plane, explain that "Crossing drops" are the region under the curve $x = \frac{d}{2}\sin(\theta)$.

First integrate from 0 to $\frac{d}{2}\sin(\theta)$:

$$\int_0^{\frac{d}{2}\sin(\theta)} dx = \frac{d}{2}\sin(\theta)$$

$$Then\ integrate\ the\ result\ from\ 0\ to\ \frac{\pi}{2}$$

$$Crossing\ drops = \int_0^{\frac{\pi}{2}} \frac{d}{2}\sin(\theta)d\,\theta =$$

Note: In general, if d is not equal to L the crossing condition becomes:

$$x \leq \frac{L}{2}\sin(\theta)$$

Part D — The probability result

Use the area ratio argument to show:

$$P(\text{cross}) = \frac{2}{\pi}(\text{when } L = d)$$

Explain why an experiment with Ndrops and Ccrossings gives:

$$\hat{P} = \frac{C}{N}$$

So the estimate for π is:

$$\pi \approx \frac{2}{\hat{P}} = \frac{2N}{C}$$

Part D — Do an experiment (or simulation)

Perform an experiment (throw the needle at least 20 times). Record:

$N =$______

$C =$______

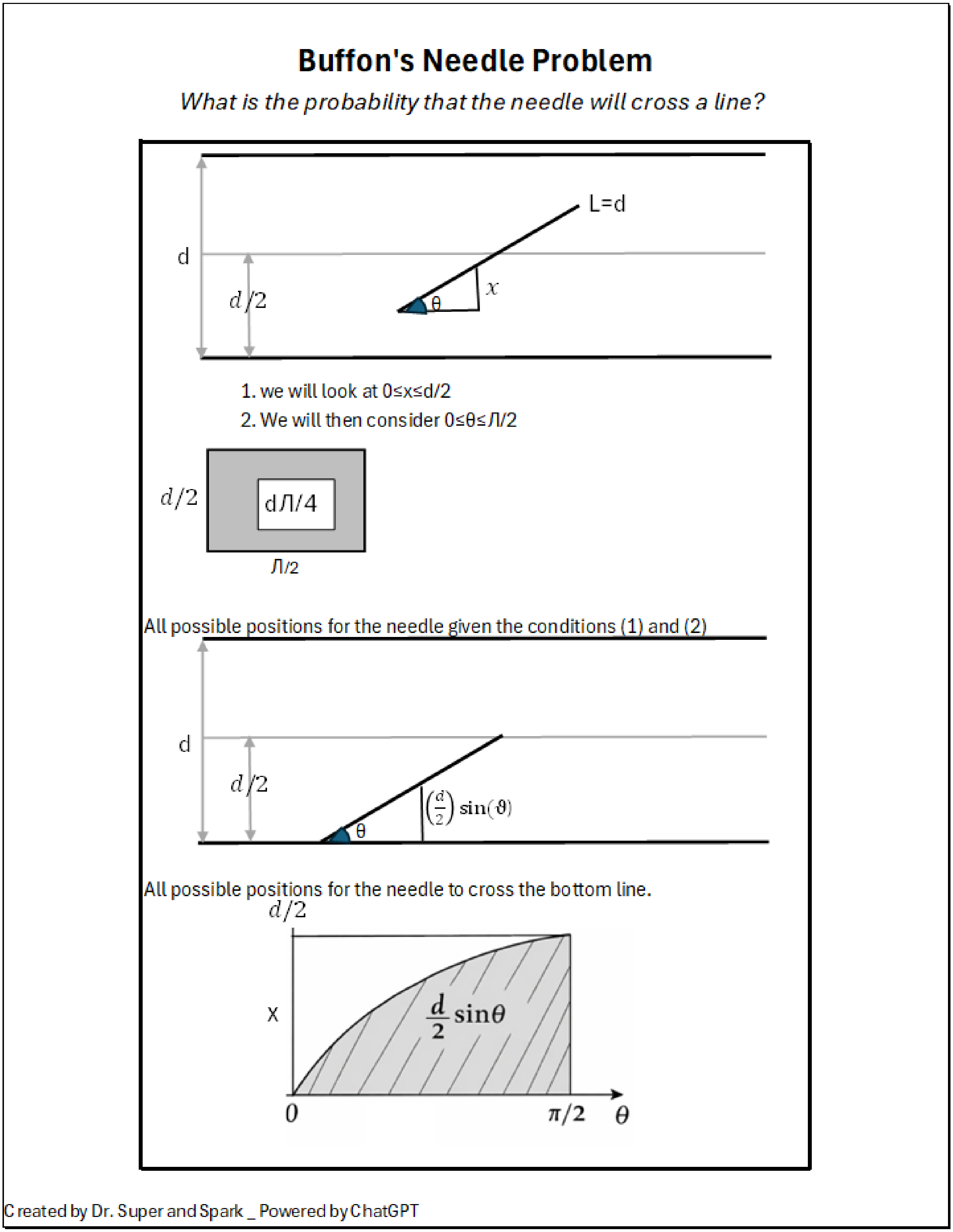
$\pi \approx \frac{2N}{C} =$______

Repeat with a larger N. Did your estimate get closer to π?

Extension (optional) – If $L \neq d$, the general formula is:

$$P(\text{cross}) = \frac{2L}{\pi d}$$

Challenge: If you double the needle length L, what happens to the probability? Explain in one sentence.


Buffon's Needle Problem
What is the probability that the needle will cross a line?
L=d
d
d/2
x
θ
1. we will look at 0≤x≤d/2
2. We will then consider 0≤θ≤Л/2
d/2
dЛ/4
Л/2
All possible positions for the needle given the conditions (1) and (2)
d
d/2
θ
(d/2) sin(ϑ)
All possible positions for the needle to cross the bottom line.
d/2
X
d/2 sinθ
0
π/2
θ
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Adventure 10 — Buffon's Needle Problem —Student Activity Sheet

Goal: Estimate π using geometry + probability.

Materials:

Paper (with parallel lines), ruler (or printed sheet), and a "needle" (toothpick)

OR use an online simulation / random generator (optional)

Setup

Parallel lines are spaced d units apart. A needle of length L is dropped at random.

For this activity we use the special case: L = d

Part A — Define the random variables

Define:

x = distance from the needle's center to the nearest line

θ = angle the needle makes with the lines

Give the ranges:

$0 \le x \le d/2$

$0 \le \theta \le \pi/2$

Part B — All possible positions for the needle given (1) and (2)

Compute the following two integrals and show that all the possible positions for the needle given (1) and (2) is d π/4 their product. You can also see that from the rectangle that has sides d/2 and π/4

First Integrate from 0 to d/2

$$\int_0^{d/2} dx =$$

(d/2)(π/2) = d π/4

$$\textit{Integrate the result from } 0 \textit{ to } \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} (\quad) d\theta =$$

Part C— When does the needle cross a line given (1) and (2)?

Use the picture and explain why the needle crosses the nearest line when L = d:

$$x \le \frac{d}{2}\sin(\theta)$$

In the (θ, x)-plane, explain that "Crossing drops" are the region under the curve $x = \frac{d}{2}\sin(\theta)$.

First integrate from 0 to $\frac{d}{2}\sin(\theta)$:

$$\int_0^{\frac{d}{2}\sin(\theta)} dx = \frac{d}{2}\sin(\theta)$$

$$\textit{Then integrate the result from } 0 \textit{ to } \frac{\pi}{2}$$

$$\textit{Crossing drops} = \int_0^{\frac{\pi}{2}} \frac{d}{2}\sin(\theta) d\,\theta =$$

Note: In general, if d is not equal to L the crossing condition becomes:

$$x \le \frac{L}{2}\sin(\theta)$$

Part D — The probability result

Use the area ratio argument to show:

$$P(\text{cross}) = \frac{2}{\pi}(\text{when } L = d)$$

Explain why an experiment with Ndrops and Ccrossings gives:

$$\hat{P} = \frac{C}{N}$$

So the estimate for π is:

$$\pi \approx \frac{2}{\hat{P}} = \frac{2N}{C}$$

Part D — Do an experiment (or simulation)

Perform an experiment (throw the needle at least 20 times). Record:

$N =$______

$C =$______

$\pi \approx \frac{2N}{C} =$______

Repeat with a larger N. Did your estimate get closer to π?

Extension (optional)

If $L < d$, the general formula is:

$$P(\text{cross}) = \frac{2L}{\pi d}$$

Challenge: If you double the needle length L, what happens to the probability? Explain in one sentence.

🦸 Adventure 11. –The Magic of Taylor Series

🧩 Purpose

Adventure 11 introduces students to **Taylor Series** as a powerful idea:

a function can be rebuilt from small pieces of information at a single point.

Students will:

- understand Taylor Series as **successive approximations**, not formulas to memorize,
- compute **cumulative sums** for
  - e^xat $x = 1$,
  - $\cos(x)$at $x = \pi/6$,
  - $\sin(x)$at $x = \pi/6$,
- compare each approximation directly to the **actual curve**,
- see how cosine and sine combine to form

$$e^{ix} = \cos x + i\sin x$$

- connect mathematics to **historical scientific prediction** through the Halley–Newton story.

This section serves as a **capstone** for the Calculus & Mechanics series, uniting limits, derivatives, curves, and approximation.

🧱 Prerequisites

Students should already be comfortable with:

- derivatives as "rate of change" (Sections 6–7),
- basic trigonometric functions,
- reading values from a graph,
- the idea of approximation (from L'Hôpital's Rule in Section 9).

No prior knowledge of Taylor Series or complex numbers is required.

🎥 Story, Context & Setup

Begin by **reading aloud the Halley–Newton story** to establish emotional and historical context

Key teaching message to emphasize verbally:

"Halley and Newton trusted small pieces of information — local changes — to predict something enormous and far away."

This idea directly parallels Taylor Series.

Now show the following video, pausing briefly to discuss as needed:

What's so special about Euler's number e?3Blue1Brown — Essence of Calculus, Chapter 5

- why adding more terms improves accuracy,
- why the approximation is best near the expansion point.

📊 Materials Needed

- Adventure 11 Student Activity Sheet (Taylor Series)
  Calculus and Mechanics Section …
- Graphs of e^x, $\sin x$, $\cos x$(already embedded on the sheet)

- Pencil, ruler, calculator (optional but helpful)

🧮 Guided Activity Flow (Teacher View)

Part 1 — Rebuilding e^x from Pieces

Students are given the **pre-computed Taylor terms** and are asked only to:

- compute **cumulative sums**,
- compare each partial sum to the value of e^1,
- locate their result on the graph of e^x.

Teaching emphasis:
Each added term is a correction that improves the approximation.

Do **not** rush this step — this is where the "magic" first becomes visible.

Part 2 — Rebuilding $\cos x$ Repeat the same process:

- cumulative sums only,
- comparison with the curve,
- visual verification.

This repetition is intentional — it reinforces confidence.

Students:

- use the provided terms,
- compute cumulative sums at $x = \pi/6$,
- compare their approximation to the graph and known value.

Key observation to guide them toward:
Some terms vanish — this is not an accident, but structure.

Part 3 — Rebuilding $\sin x$

Repeat the same process:

- cumulative sums only,
- comparison with the curve,
- visual verification.

This repetition is intentional — it reinforces confidence.

Part 4 — Powers of i and the Final Synthesis

Students fill in:

$$1,\ i,\ -1,\ -i,\ \ldots$$

Then:

- observe how even powers behave like cosine terms,
- odd powers behave like sine terms,
- **add the two series together** to form e^{ix}.

End by pointing to the **unit circle diagram** and letting students verify:

- real part → cosine,
- imaginary part → sine.

No heavy complex-number theory is needed.

🧠 Core Teaching Ideas to Emphasize

- Taylor Series works because functions are smooth.
- Local information (value + derivatives) predicts nearby behavior.
- More terms → better approximation.
- This is exactly how Newton predicted orbits and Halley predicted a comet.

🧠 Reflection Prompt

- Use one of the following in discussion or writing:
- Halley trusted mathematics to predict something he would never live to see.
- How is building a Taylor Series similar to making that prediction?
- Why does adding more terms make the prediction better?

🧑‍🏫 Teacher Notes & Tips

- Let students own the cumulative sums — do not rescue too early.
- Encourage estimation rather than perfection.
- Rooz-type students may move fast; slow them by asking why each step improves accuracy.
- Marc-type students benefit from repeated visual confirmation on the graph.
- Emphasize that nothing mystical is happening — only careful accumulation.

🪄 Wrap-Up

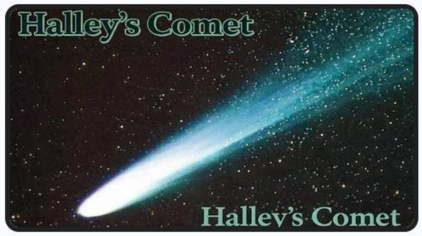
By the end of Adventure 11 students understand that:

- Taylor Series builds a function from small, understandable pieces.
- Approximation is a strength, not a weakness, of mathematics.
- The same idea predicts comets, orbits, and curves.
- Euler's formula is not magic — it is structure revealed.

This section closes the Calculus & Mechanics series with **unity, beauty, and meaning**.

🌙 **Adventure 11 Story. – Halley's Comet, Newton's Falling Moon & the Secret of Taylor Series**

Sir Isaac Newton



Halley's Comet

Sir Edmond Halley

When the kids were talking about space, I told them something I had never shared in any math class before. I said, quietly: "Thirty-eight years from now, Halley's Comet will return. I won't be here anymore. But you will be. And when you look up and see that pale streak across the sky, I want you to remember this moment — our Math Circle — and think of me." The room went still. Cyrus was emotional, Rooz and Marc stopped talking and moving. We had a n emotional moment together.

Long before rockets, long before NASA, long before we could track anything with computers, there lived a curious man named Edmond Halley. He wasn't rich. He wasn't a superstar scientist like Newton. But he had one extraordinary gift: he could look at scattered, messy data and find patterns that no one else could see. In the early 1700s he noticed something peculiar — a bright comet seen in 1682 looked very much like ones recorded in 1607 and 1531.

Astronomers told him he was wasting his time: *"You can't compare events 75 years apart"; "The orbits are different"; "The math is impossible."* Halley ignored them. After months of calculation, he made one of the boldest statements in scientific history: "They are the same comet. It returns every 76 years. It will be back in 1758." This was the first time any human predicted a cosmic event so far into the future — and he was right.

But Halley did not do it alone. He had a secret weapon: Isaac Newton. Newton had just discovered something shocking — something that changed how humans understood the universe. He realized that the same force that pulls an apple down also pulls the Moon toward the Earth. The Moon is *falling*, but because it keeps missing the Earth as it falls, it stays in orbit. When Halley asked Newton to explain how comets move, Newton used an idea that was revolutionary for its time: you can understand a complicated path by breaking it into tiny steps.

Start with gravity here; Measure a small change there; Approximate the motion step by step;
Tiny steps → Giant arcs; Small formulas → Predictions across the sky.

This simple, brilliant idea — using small changes to understand big curves — eventually grew into what we now call **Taylor Series**. It is the mathematics of "little steps," the same idea that let Halley predict a comet and Newton map the orbit of the Moon.

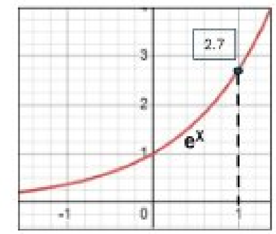
And now, in Section 10, *you* will use that same idea. Not to find comets. Not to save astronauts. But to understand something just as magical: How small pieces of information can reveal the shape of an entire function.

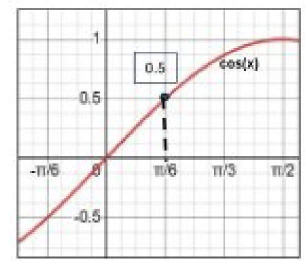
And maybe — just maybe — when Halley's Comet returns in 38 years, you will look up at the night sky and whisper, "We learned this with Dr. Super."

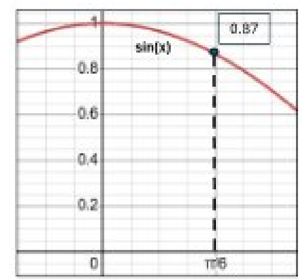
Adventure 11 – Student Activity Sheets Solutions

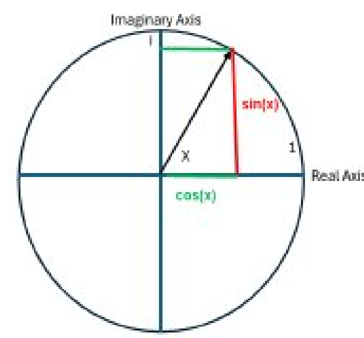
The Magic of Taylor Series - Student Activity Solution

Taylor Series for f(X)	f(0)	f'(0)	f''(0)	f'''(0)	f''''(0)	f'''''(0)	f''''''(0)	
	x	x	x	x	x	x	x	
Around X=0	1 +	X/1 +	X2/2! +	X3/3! +	X4/4! +	X5/5! +	X6/6! +	...
	e^0	e^0	e^0	e^0	e^0	e^0	e^0	
	1	1	1	1	1	1	1	
Taylor Series for e^X	1 +	X/1 +	$X^2/2!$ +	$X^3/3!$ +	$X^4/4!$ +	$X^5/5!$ +	$X^6/6!$ +	...
X=1	1.000 +	1.000 +	0.500 +	0.167 +	0.042 +	0.008 +	0.001 +	...
Cumulative value	1.000 =>	2.000 =>	2.500 =>	2.667 =>	2.708 =>	2.717 =>	2.718	...
e^1=2.718								
	cos(0)	-sin(0)	-cos(0)	sin(0)	cos(0)	-sin(0)	-cos(0)	
	1	0	-1	0	1	0	-1	
Taylor Series for cos(X)	1 +	0 +	-X2/2 +	0 +	X4/4! +	0 +	-X6/6! +	...
X=Л/6	1.000 +	0.000 +	-0.548 +	0.000 +	0.050 +	0.000 +	-0.002	
Cumulative value	1.000 =>	1.000 =>	0.452 =>	0.452 =>	0.502 =>	0.502 =>	**0.500**	
cos(Л/6)=0.5								
	sin(0)	cos(0)	-sin(0)	-cos(0)	sin(0)	cos(0)	-sin(0)	
	0	1	0	-1	0	1	0	
Taylor Series for sin(X)	0 +	X/1 +	0 +	-X3/6 +	0 +	X5/5! +	0 +	...
X=Л/6	0.000 +	1.047 +	0.000 +	-0.191 +	0.000 +	0.010 +	0.000	
Cumulative value	0.000 =>	1.047 =>	1.047 =>	0.856 =>	0.856 =>	0.866 =>	**0.866**	
sin(Л/6)=0.866								
	1	i	i^2	i^3	i^4	i^5	i^6	
	1	i	-1	-i	1	i	-1	
Taylor Series for e^{iX}	1 +	X/1 +	X2/2 +	X3/6 +	X4/4! +	X5/5! +	X6/6! +	...
	1 +	iX/1 +	-X2/2 +	-iX3/6 +	X4/4! +	iX5/5! +	-X6/6! +	...
cos X	1 +	0 +	-X2/2 +	0 +	X4/4! +	0 +	-X6/6! +	...
isinX	0 +	iX/1 +	0 +	-iX3/6 +	0 +	iX5/5! +	0 +	...
e^{iX}=cosX+isinX	1 +	iX/1 +	-X2/2 +	-iX3/6 +	X4/4! +	iX5/5! +	-X6/6! +	...









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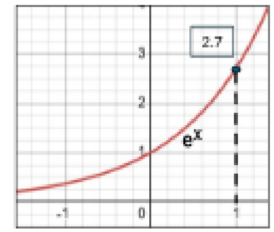
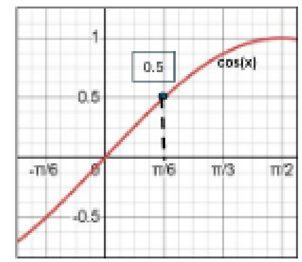
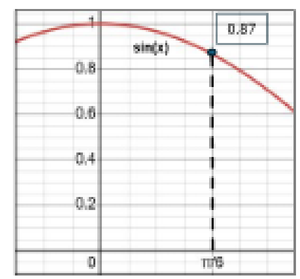
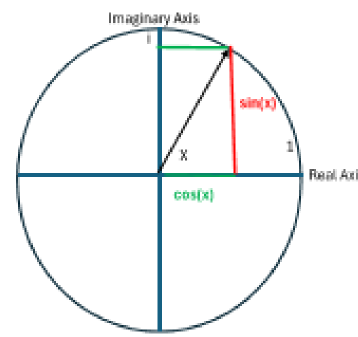
Taylor Series for f(X) Around X=0	f(0) x 1	+ f'(0) x X/1	+ f''(0) x $X^2/2!$	+ f'''(0) x $X^3/3!$	+ f''''(0) x $X^4/4!$	+ f'''''(0) x $X^5/5!$	+ f''''''(0) x $X^6/6!$	+ ...
	e^0	e^0	e^0	e^0	e^0	e^0	e^0	
Fill in the Values ==>								
Taylor Series for e^X	1	+ X/1	+ $X^2/2!$	+ $X^3/3!$	+ $X^4/4!$	+ $X^5/5!$	+ $X^6/6!$	+ ...
X=1	1.000	+	+	+	+	+	+	+ ...
Compute Cumulative value ==>		=>	=>	=>	=>	=>	=>	...
e^1=2.718								
	cos(0)	-sin(0)	-cos(0)	sin(0)	cos(0)	-sin(0)	-cos(0)	
Fill in the Values ==>								
Taylor Series for cos(X)	1	+ 0	+ $-X^2/2$	+ 0	+ $X^4/4!$	+ 0	+ $-X^6/6!$	+ ...
X=Л/6	1.000	+	+	+	+	+	+	
Compute Cumulative value ==>		=>	=>	=>	=>	=>	=>	
cos(Л/6)=0.5								
	sin(0)	cos(0)	-sin(0)	-cos(0)	sin(0)	cos(0)	-sin(0)	
Fill in the Values ==>								
Taylor Series for sin(X)	0	+ X/1	+ 0	+ $-X^3/6$	+ 0	+ $X^5/5!$	+ 0	+ ...
X=Л/6		+	+	+	+	+	+	
Compute Cumulative value ==>		=>	=>	=>	=>	=>	=>	
sin(Л/6)=0.866								
	1	i	i^2	i^3	i^4	i^5	i^6	
Fill in the Values ==>								
Taylor Series for e^X		+	+	+	+	+	+	+ ...
Taylor Series for e^{iX}		+	+	+	+	+	+	+ ...
cos X		+	+	+	+	+	+	+ ...
+isinX		+	+	+	+	+	+	+ ...
e^{iX} = [ ] Fill Values ==>		+	+	+	+	+	+	+ ...

The Magic of Taylor Series - Student Activity Sheet









⭐ Adventure 11 – The Magic of Taylor Series — Student Activity Guide

Today you will discover **how curves can be rebuilt from tiny pieces** — one little term at a time.. **Your job is to add these pieces, step by step, and see how the curve will appear.**

You will complete four mini-activities:

1. e^xat $x = 1$
2. $\cos X$at $X = \pi/6$
3. $\sin X$at $X = \pi/6$
4. Combining $\cos X + i \sin X$ to see what you get.

Use the **cumulative sum boxes** to track how each partial sum gets closer to the real curve value.

Part 1 — Rebuilding $e^x$from its Taylor Series and Computing e^1

In the **first big block** of the table (Taylor Series for e^x), the derivatives are computed as $\mathbf{e^0}$fill in the row of derivatives at 0:

✓ Your task: Write each value in the boxes under $f(0), f'(0), f''(0), \ldots$.

On your sheet, you see the Taylor Series for e^xaround $x = 0$: $\boldsymbol{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$

For $x = 1$, the values of the individual terms have already been computed for you.

✓ Your task: Add each term to the running total (cumulative sum); After each addition, write the new cumulative value in the arrow boxes, Compare your final value with the value of $\boldsymbol{e^1 \approx 2.718}$ that is given.

✓ Look at the top-right graph:

On the e^x**curve** (first graph on your page), locate the point at $x = 1$.

Check: **Does your final cumulative sum match the height of the curve?**

This shows that a Taylor Series really does "build" the curve.

Part 2 — Rebuilding $\cos X$ from its Taylor Series

Your sheet gives the pattern of the derivatives at 0:

✓ Your task: Write each value in the boxes under $f(0), f'(0), f''(0), \ldots$.

These become the terms: $\boldsymbol{\cos X = 1 - \frac{X^2}{2!} + \frac{X^4}{4!} - \frac{X^6}{6!} + \cdots}$

For $X = \pi/6$, all the individual term values are already provided.

✓ Your task: Add the terms one by one in the cumulative sum row.

- Watch how the approximation jumps from 1 → 1 → …
- Compare the final value to $\cos(\pi/6) = 0.5$.

✓ Check the middle-right graph: On the **cosine curve**, locate $\pi/6$, verify that the Taylor approximation approaches the actual curve value.

Part 3 — Rebuilding $\sin X$ from its Taylor Series

Your sheet gives the pattern of the derivatives at 0:

✔ **Your task:** Write each value in the boxes under $f(0), f'(0), f''(0)$, ….

Which leads to: $\mathbf{\sin X = X - \frac{X^3}{3!} + \frac{X^5}{5!} - \cdots}$
Again, the term values for $X = \frac{\pi}{6}$ are already given.

✔ **Your task:** Add the terms using the cumulative boxes, Compare your final number with $\sin(\pi/6) =$ 0.866.

✔ **Check the bottom-right graph:** Find the point for $\sin X$ at $X = \pi/6$.

See how your cumulative sums steadily approach the real value.

Part 4 — Bringing Everything Together: e^{iX}

Now look at the final part of your sheet.

You see the Taylor Series for: $e^{iX}, \cos X, i\sin X$

The derivative pattern cycles through: i, i^2, i^3, i^4, i^5, i^6, Fill in these values below them in the boxes that are provided. This produces alternating real and imaginary pieces — exactly matching the series for cosine and sine.

✔ **Your task:** Add the provided terms for cos(x) and is in(x) and compare it with Taylor Series for e^{iX}:

✔ **Check the unit circle diagram:**
The final point on the complex plane should sit at:

$$(\cos X, \sin X)$$

This shows visually that:

$$\mathbf{e^{iX} = \cos X + i\sin X}$$

When your cumulative sums match the coordinates on the circle, you have **reconstructed Euler's Formula** from scratch.

What You Learned

- A Taylor Series builds a function from many tiny pieces.
- By adding more terms, your approximation gets closer to the real curve.
- Sine, cosine, and exponential all "grow" from the same idea.
- And when we extend the exponential to imaginary numbers, the cosine and sine pieces combine perfectly to form a point on the unit circle.

You have now learned the same idea that Newton, Euler, and Halley used to predict motion across the sky — from falling moons to returning comets.

🦸 Adventure 12. – Maximum Height, Critical Points, and Vertical Motion

🎯 Purpose of Adventure 12

Adventure 12 introduces students to the idea of a **maximum (critical point)** using motion they already understand:

- a **ball thrown straight up**
- a **cannonball fired vertically**
- a **cannonball fired at an angle**

Although these motions look different, students discover that their **vertical motion is identical** when the **initial vertical speed is the same**.

This section builds a **physical intuition** for the calculus idea that:

A maximum occurs when the **velocity is zero**, even though **acceleration is not**.

🎥 Watch First (5 minutes)

Watch: Introduction to Projectile Motion (Sabins)

Before starting the activity, ask students to watch with one guiding question:

What happens to the vertical velocity at the highest point of the motion?

You are not asking for formulas — only **observations**.

📜 Read or Listen Next

Story: Cannonball *and the Moment That Changed Motion*

This story provides:

- historical grounding (Galileo → Newton),
- narrative motivation,
- and a natural transition from **motion** to **mathematical thinking**.

Encourage students to listen to:

- the *pause* at the top,
- the idea that motion can be understood in one direction at a time.

🧭 What Students Will Do

In the student activity, students will:

- Read **Distance–Velocity–Acceleration (DiVA)** graphs
- Identify the **maximum height**
- Connect the maximum height to **velocity = 0**
- Compare:
    - a straight-up throw
    - a vertical cannon shot
    - an angled cannon shot

- Extend the idea to **range** (horizontal distance)

No advanced calculus is required — this is **conceptual groundwork**.

📊 The Three Core Visuals (Use in Order)

Page 1 — Ball Thrown Straight Up

Key observations to guide students toward:

- Height reaches **45 m at 3 seconds**
- Velocity crosses **zero** at exactly the same time
- Acceleration stays **constant and negative**

Teacher emphasis:

"Notice: the ball stops going *up*, not because acceleration is zero, but because velocity is."

Page 2 — Cannon Ball Vertical Flight

Key insight:

- Despite a very different *launch*, the **graphs are identical** vertically.

Ask:

- *What changed physically?* (how the ball was launched)
- *What did NOT change mathematically?* (vertical motion)

This reinforces:

Vertical motion depends only on **initial vertical speed** and **gravity**.

Page 3 — Same Vertical Motion, Different Paths

This page is the conceptual payoff.

Students see:

- a **parabolic path in space**
- the **same height-vs-time curve**
- the **same velocity-vs-time graph**

Teacher framing:

"Different paths. Same vertical story."

This prepares students for:

- projectile motion,
- independence of horizontal and vertical motion,
- and later optimization problems.

⭐ The Big Idea (Make This Explicit)

By the end of the activity, students should be able to say:

A **maximum height** occurs when the **vertical velocity is zero**,
even though the **acceleration remains constant**.

This is the **physical meaning** of a **critical point**.

🧠 Suggested Discussion Questions

Use one or two — not all.

- Why doesn't the ball keep going up if acceleration is never zero?
- How can two objects follow the same vertical motion but land in different places?
- Which graph tells you *when* the maximum occurs?
- Which graph tells you *why* it occurs?

🧭 Optional Extension

If students are curious or advanced:

- Show the **angled cannon range**
- Ask:

"What controls how far it goes?"

- Let them discover:
  - time in air comes from **vertical motion**
  - distance comes from **horizontal speed**

You are planting seeds for **optimization** later.

🧩 Where This Fits in the Course

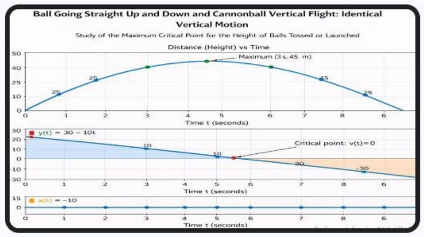
Section 11 bridges:

- **motion (physics)**
- **graphs (functions)**
- **calculus ideas (maxima, critical points)**

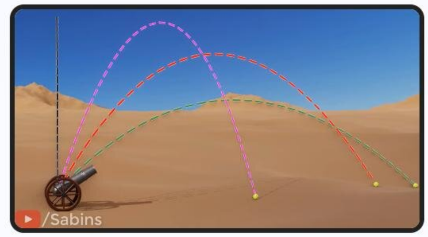
It prepares students for: derivatives as rates of change, optimization problems, and real-world modeling.

📜 Adventure 12 Story – The Cannonball & the Moment That Changed Motion

Ball Toss: Straight Up and Straight Down



Vertical Motion Graphs: Height and Vertical Velocity



Cannon Paths Look Different, but the Vertical Part Matches the ball going up and down

In the early 1600s, European battlefields thundered with cannons. Iron balls flew through smoke-filled skies — powerful, dangerous, and wildly unpredictable. Gunners aimed, fired, and hoped. No one could say where a cannonball would land or why. Yet every shot revealed the same quiet mystery. The cannonball always rose, slowed, paused for the briefest instant — and then fell. That pause changed science forever.

⚔️ **From Battlefields to Questions:** In **Italy**, a young mathematician named **Galileo Galilei** watched cannonballs and falling stones with a new kind of curiosity. Instead of asking *where* they landed, he asked: **How does motion change from moment to moment?** Galileo realized that guessing was useless. Motion had to be measured. He focused on one direction at a time — especially **up and down**.

🎯 **The Vertical Secret:** The instant a cannonball leaves the cannon, gravity begins pulling it downward. Every second: the upward velocity decreases by the same amount; the ball slows smoothly; until, at one exact moment, the upward motion reaches zero. At that instant, the cannonball is higher than it will ever be. This moment is not luck. It is inevitable.

If a ball is thrown straight up with the same vertical speed that the cannonball has when it leaves the cannon, then both objects follow *exactly the same vertical motion*. They rise for the same amount of time, reach the same maximum height, pause, and fall back to the ground in the same way. The cannonball may travel forward across the sky, but vertically, it behaves no differently than a ball tossed straight up by hand.

📈 **Newton Finds the Peak:** A generation later, in **England**, **Isaac Newton** gave this moment a name and a language. Height, Newton showed, is a function of time. Velocity is the *slope* of that height. Acceleration is what changes the slope when velocity becomes zero, the slope flattens. That flat point is the **maximum** — the top of the curve — what we now call a **critical point**.

🚀 **Why This Changed the World:** Once this idea was understood, cannons stopped being guesses. Engineers could finally predict how high a ball would rise; when it would stop climbing; how far it would travel. The same mathematics now guides rockets, satellites, and space probes. It governs basketball shots, golf swings, and planetary motion. Every arc in the sky carries the same hidden structure.

🧠 **The Big Idea:** The most important moment in motion is not when something moves the fastest. It is the instant when **change pauses**. That pause is where calculus lives. And that is why, on your DiVA charts, the highest point of the height curve lines up exactly with zero of the velocity curve. The cannonball didn't just reshape warfare. It taught humanity how to understand change itself.

🎯 Adventure 12 - Student Activity Solutions

From a Thrown Ball to a Cannon Ball: Understanding the Maximum

In this activity, you will study two motions:

1. A **ball thrown straight up**
2. A **cannon ball fired from a cannon**

Although they look different, their **vertical motion is the same**. Your goal is to discover **where the maximum occurs** and what it means.

🔹 Part 1: A Ball Thrown Straight Up

The height of a ball (in meters) as a function of time (in seconds) is shown on the **DiVA charts**.

1. Reading the Height Graph

Look at the **Distance (Height) vs Time** graph.

- At what time does the ball reach its **greatest height**?
  $t = \mathbf{3}$ seconds
- What is the **maximum height** of the ball?
  $y = \mathbf{45}$ meters

Find the highest point on the curve. This point is called the **maximum**.

2. Reading the Velocity Graph

Now look at the **Velocity vs Time** graph.

- What is the (vertical) velocity of the ball at t=0 $v(start) = \mathbf{30}\ meters/$second
- What is the (vertical) velocity of the ball at t=6 $v(end) = \mathbf{-30}\ meters/$second
- What is the relation between $v(start) and\ v(end)$ $\boldsymbol{one\ is\ the\ opposite\ of\ the\ other}$
- At what time does the velocity equal **zero**? $t = 3$ seconds
- What is the ball doing at that instant? ☐ moving upward **X** not moving up or down ☐ moving downward

3. Connecting Height and Velocity

Complete the sentence:

The ball reaches its **maximum height** at the moment when the **velocity is: 0**

🔹 Part 2: The Acceleration Clue

Look at the **Acceleration vs Time** graph.

- What is the value of the acceleration during the entire motion?

$$a = -10\ \text{m/s}^2$$

- Does the acceleration ever become zero?
- ☐ yes **X** no

Even though acceleration is constant, the velocity still changes. Write one sentence explaining why:

The acceleration of gravity is constant and will slow down the velocity of the ball and then give it the negative velocity.

🔹 **Part 3: A Cannon Ball Fired from a Cannon**

Now imagine the same vertical motion, but the ball comes out of a **cannon**.

Look at the **Cannon Ball Vertical Flight** DiVA charts.

- Vertical velocity of the cannon ball at t=0 is: . $v(fire) = \mathbf{30}\ meters/$second
- Maximum height of the cannon ball: $y = 45 \quad meters$
- Time when the cannon ball stops rising: $t = 3$ seconds

Complete the statement: Even though one object is **thrown by a very strong hand** and the other is **fired from a cannon,** their **vertical velocity is: always equal to each other.**

🔹 **Part 4: How Far the Cannon Ball Travels**

The cannon ball also moves **sideways**.

From the height graph: Total time in the air: $t = 6$ seconds

Suppose the cannon ball moves horizontally at a constant speed of:

$$v_x = 30 \text{ m/s}$$

Calculate how far it travels (range) before hitting the ground: distance traveled $=$ speed $\times$ time

Range $= \mathbf{30} \times \mathbf{6} \quad = \mathbf{180}$ meters

- Check the range on the chart that shows the flight of the cannon ball when it is fired at an angle.
- The Range on the chart is the same as what I found here: **X Yes** ☐ No

⭐ **The Big Idea**

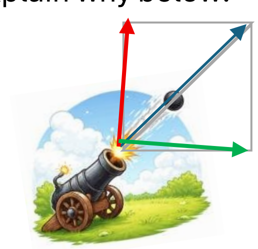
Complete this carefully:

A **maximum height** occurs when the **vertical velocity is zero,** even though the **acceleration remains constant.**

🧠 **Reflection**

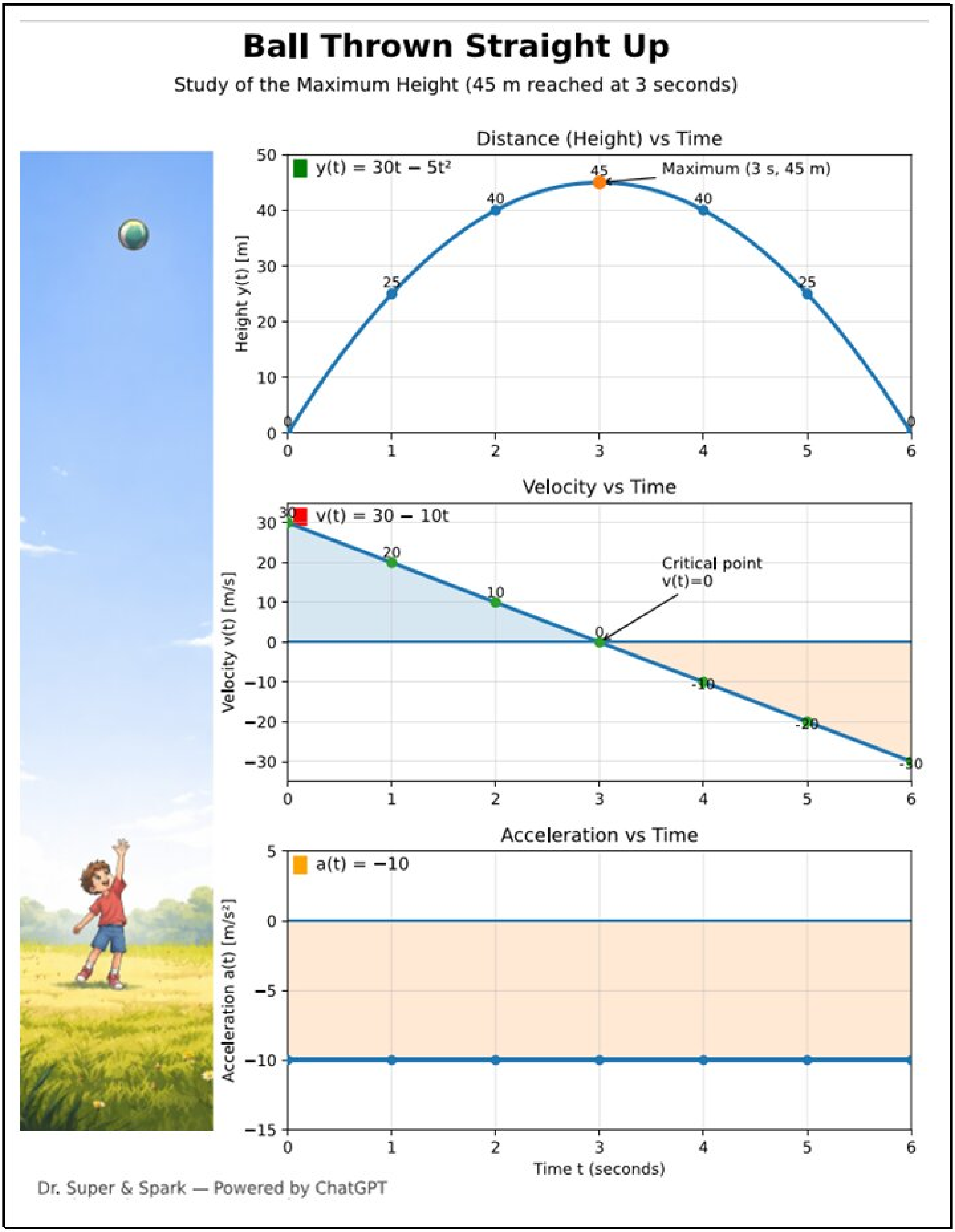
The **maximum height** depends only on the **vertical velocity** but the **distance traveled** depends on the **horizontal velocity**?

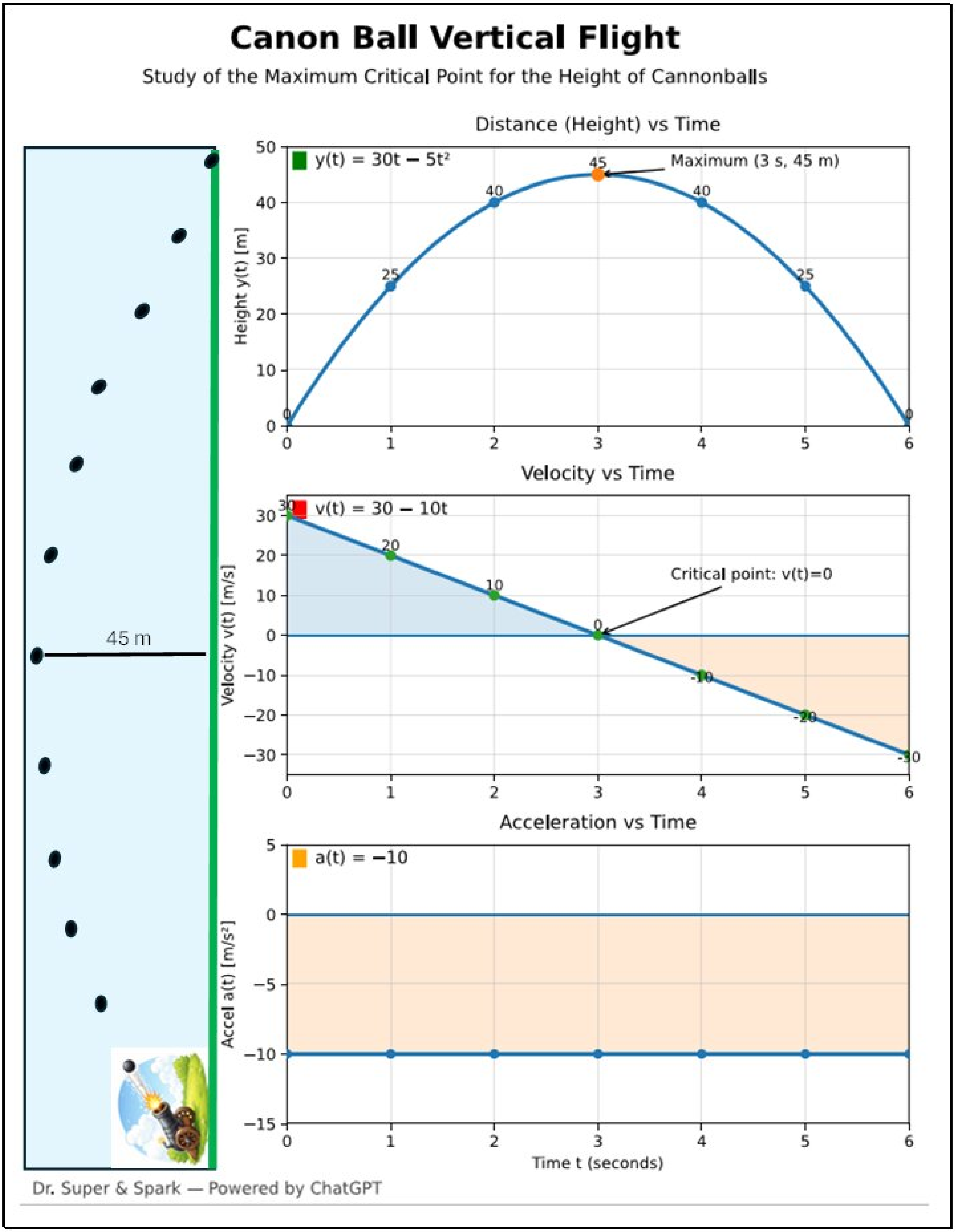
For this cannonball the **vertical** and **horizontal** velocities were both equal to **30 m/s**, so the angle that the cannonball was fired at is 45 degrees. Explain why below:

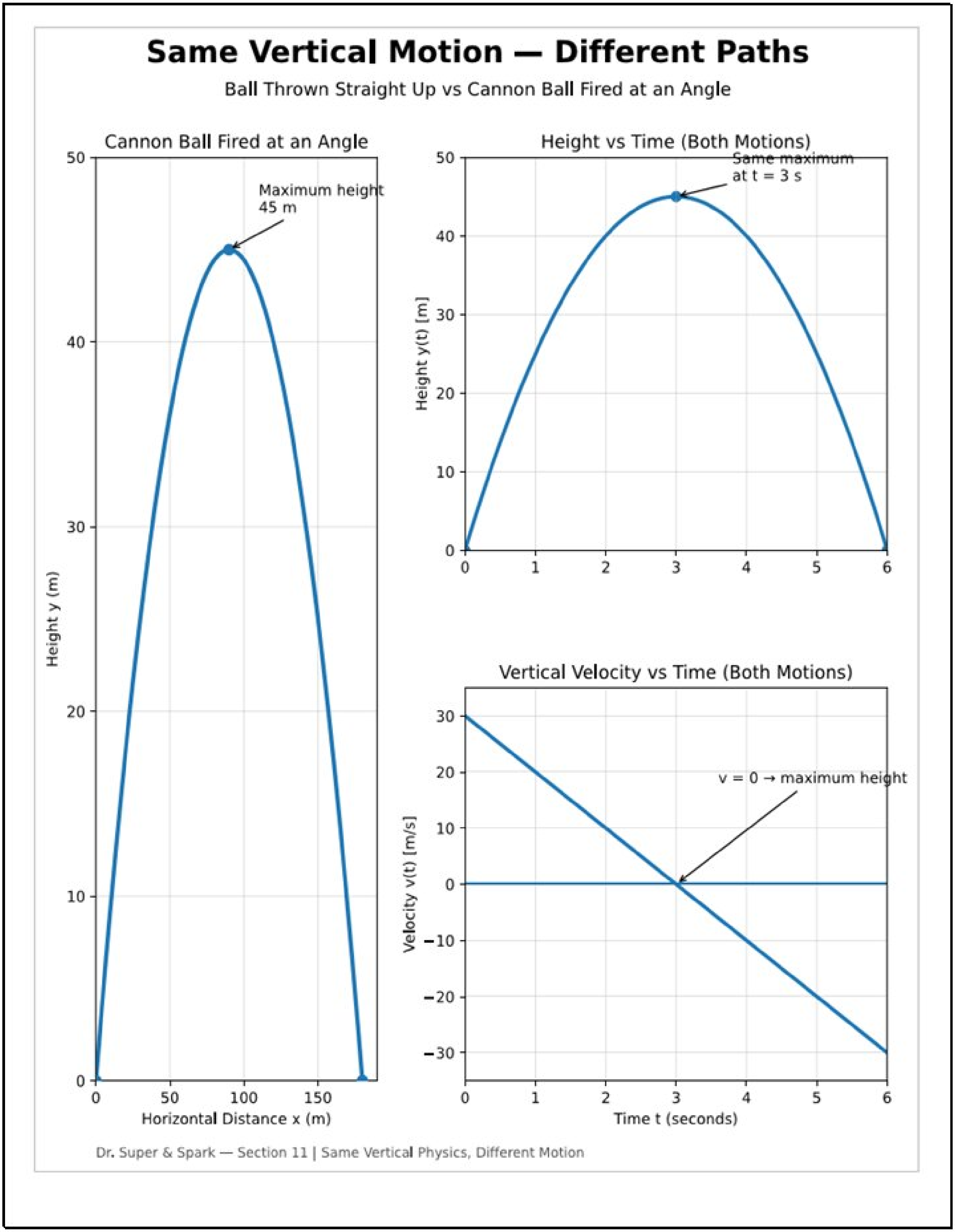


Vertical velocity = sin(θ) = Horizontal velocity = cos(θ) since sin(θ)=cos(θ) then θ=45 degrees or Л/4.

🎯 Adventure 12 - Student Activity Charts




Canon Ball Vertical Flight
Study of the Maximum Critical Point for the Height of Cannonballs
45 m
Distance (Height) vs Time
y(t) = 30t − 5t²
Maximum (3 s, 45 m)
Height y(t) [m]
Velocity vs Time
v(t) = 30 − 10t
Critical point: v(t)=0
Velocity v(t) [m/s]
Acceleration vs Time
a(t) = −10
Accel a(t) [m/s²]
Time t (seconds)
Dr. Super & Spark — Powered by ChatGPT


Same Vertical Motion — Different Paths
Ball Thrown Straight Up vs Cannon Ball Fired at an Angle
Cannon Ball Fired at an Angle
Maximum height
45 m
Height y (m)
Horizontal Distance x (m)
Height vs Time (Both Motions)
Same maximum
at t = 3 s
Height y(t) [m]
Vertical Velocity vs Time (Both Motions)
v = 0 → maximum height
Velocity v(t) [m/s]
Time t (seconds)
Dr. Super & Spark — Section 11 | Same Vertical Physics, Different Motion

🎯 Section 12 - Student Activity Sheet

From a Thrown Ball to a Cannon Ball: Understanding the Maximum

In this activity, you will study two motions:

3. A **ball thrown straight up**
4. A **cannon ball fired from a cannon**

Although they look different, their **vertical motion is the same**. Your goal is to discover **where the maximum occurs** and what it means.

🔹 Part 1: A Ball Thrown Straight Up

The height of a ball (in meters) as a function of time (in seconds) is shown on the **DiVA charts**.

1. Reading the Height Graph

Look at the **Distance (Height) vs Time** graph.

- At what time does the ball reach its **greatest height**?
  $t =$ _______ seconds
- What is the **maximum height** of the ball?
  $y =$ _______ meters

Find the highest point on the curve. This point is called the **maximum**.

2. Reading the Velocity Graph

Now look at the **Velocity vs Time** graph.

- What is the (vertical) velocity of the ball at t=0 $v(start) =$ _________ $meters/$second
- What is the (vertical) velocity of the ball at t=6 $v(end) =$ _________ $meters/$second
- What is the relation between $v(start) and\ v(end)$ ____
- At what time does the velocity equal **zero**? $t =$ _______ seconds
- What is the ball doing at that instant? ☐ moving upward ☐ not moving up or down ☐ moving downward

3. Connecting Height and Velocity

Complete the sentence:

The ball reaches its **maximum height** at the moment when the **velocity is:** ________________.

🔹 Part 2: The Acceleration Clue

Look at the **Acceleration vs Time** graph.

- What is the value of the acceleration during the entire motion?

$$a = \_\_\_\_\_\_\_\_\_\_ \text{m/s}^2$$

- Does the acceleration ever become zero?
- ☐ yes ☐ no

Even though acceleration is constant, the velocity still changes. Write one sentence explaining why:

____

◆ Part 3: A Cannon Ball Fired from a Cannon

Now imagine the same vertical motion, but the ball comes out of a **cannon**.

Look at the **Cannon Ball Vertical Flight** DiVA charts.

- Vertical velocity of the cannon ball at t=0 is: . $v(fire) =$ $meters/$second
- Maximum height of the cannon ball: $y =$ meters
- Time when the cannon ball stops rising: $t =$ seconds

Complete the statement: Even though one object is **thrown by a very strong hand** and the other is **fired from a cannon,** their **vertical velocity is:** ______________________________

◆ Part 4: How Far the Cannon Ball Travels

The cannon ball also moves **sideways**.

From the height graph: Total time in the air: $t =$ seconds

Suppose the cannon ball moves horizontally at a constant speed of:

$$v_x = 30 \text{ m/s}$$

Calculate how far it travels (range) before hitting the ground: distance traveled $=$ speed $\times$ time

Range $= 30 \times$ ___ $=$ _______ meters

- Check the range on the chart that shows the flight of the cannon ball when it is fired at an angle.
- The Range on the chart is the same as what I found here: ☐ Yes ☐ No

⭐ The Big Idea

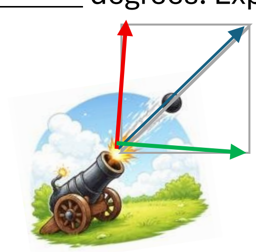
Complete this carefully:

A **maximum height** occurs when the **vertical velocity is** ___________, even though the **acceleration remains** __________________________

🧠 Reflection

The **maximum height** depends only on the **vertical** ___________ but the **distance traveled** depends on the **horizontal** _____________?

For this cannonball the **vertical** and **horizontal** velocities were both equal to __________ m/s, so the angle that the cannonball was fired at is _______________ degrees. Explain why below:



🧑‍🏫 Adventure 13 – Drawing Curves – Maximum, Minimum and Inflection Points

🧭 Purpose of Adventure 13

This section teaches students how to construct a curve using calculus.

Students learn:

- First derivative → finds critical points
- Second derivative → classifies them
- Signs between zeros → determine behavior
- A full curve can be reconstructed logically

This is a conceptual turning point in Calculus & Mechanics.

🎯 Learning Objectives

By the end of this section students will:

1. Compute first and second derivatives of a cubic function.
2. Identify zeros of derivatives.
3. Classify maxima, minima, and inflection points.
4. Construct a sign table.
5. Sketch a function using calculus reasoning alone.
6. Connect biological interpretation to mathematical structure.

🧱 Prerequisites

Students should already understand:

- Derivatives as slope
- Second derivative as concavity
- Quadratic and cubic shapes
- Basic sign analysis

📖 Story Integration

Begin with the ATP narrative.

Key teaching line:

"Anything that flows over time has a curve."

ATP becomes the biological analogue of position in mechanics.

Do not rush this part. It anchors the abstraction.

🧮 Teaching Sequence

🔹 Part 1 – Derivatives

Students compute:

- $D'(t)$

- $D''(t)$

Pause and ask:

- What does the first derivative mean biologically?
- What does the second derivative mean?

Make them articulate:

- V = rate of ATP change
- A = change of the rate

◆ **Part 2 – Draw Acceleration First**

Students graph A(t).

Why first?

Because it is linear and easiest to interpret.

Students identify:

- Intercept
- Zero
- Slope

Ask: What happens to V where A = 0?

This creates predictive reasoning.

◆ **Part 3 – Draw Velocity**

Students:

- Find roots
- Identify t = 10 and t = 30
- Identify slope zero at t = 20

Ask: Without seeing D yet, what must happen at t = 10 and t = 30?

Students should answer: critical points.

◆ **Part 4 – Sign Table**

This is the structural heart.

Students fill:

- Signs of V
- Signs of A
- Behavior of D

Reinforce:

Signs between zeros determine direction.

◆ **Part 5 – Sketch D(t)**

Students now sketch using:

- Values at 0, 10, 20, 30, 40
- Signs of derivatives
- Classification of critical points

Only after they complete this should the DiVA chart be shown.

🧠 Activity 3 – Modified ATP Model

Students analyze a model with early and late energy boosts.

This reinforces:

- Same mathematics
- Different story

Emphasize transfer of structure.

💡 Core Teaching Insights

1. First derivative finds where slope is zero.
2. Second derivative determines concavity.
3. Zero slope does not determine type.
4. You can sketch without plotting every point.

This is advanced conceptual reasoning for high school students.

🪄 The "Parting Shot"

Ask:

If you had only the zeros of V and A and the signs between them, could you draw the entire ATP curve?

The answer is yes.

This is powerful because:

- It shows structure controls shape.
- It reduces graphing to reasoning.
- It unifies biology and mechanics.

🧑‍🏫 Teacher Tips

- Encourage verbal reasoning before drawing.
- Insist on justification for max/min.
- Slow down at the inflection point.
- Use color coding when possible.

🧠 Reflection Questions

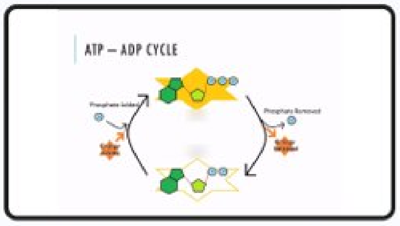
- Why is the second derivative necessary?
- Why are signs more important than exact numbers?
- How does this apply to motion problems?

🏁 Closing Thought

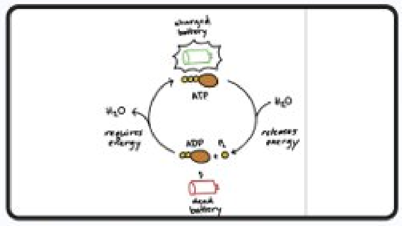
Adventure 13 marks the moment where students stop seeing derivatives as procedures and begin seeing them as structural tools.

This is the intellectual transition from computational calculus to conceptual calculus.

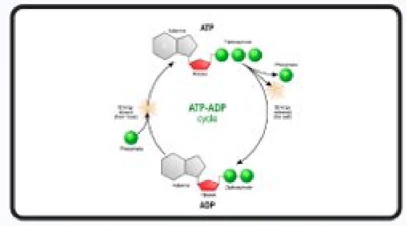
Adventure 13 Story – Drawing Curves: ATP - The Curve Behind Every Movement



ATP–ADP Cycle: Storing and Releasing Energy



ATP as a Rechargeable Energy Battery



ATP–ADP Cycle Inside the Cell

Every movement you have ever made — walking, jumping, throwing a ball, even breathing — has been powered by the same invisible molecule. Not food. Not calories. Not energy in the abstract. A specific molecule called **ATP**, adenosine triphosphate. ATP is the immediate currency of motion inside your body. When a muscle contracts, ATP is spent. When ATP runs low, motion slows or stops. Long before calculus entered textbooks, ATP was already tracing curves inside living cells.

ATP was identified in the late 1920s, when scientists were trying to answer a deceptively simple question: *what actually powers muscle contraction?* Working independently, Karl Lohmann, and soon after Cyrus Fiske and Yellapragada Subbarow, isolated a molecule that appeared whenever work was done inside muscle tissue. What made ATP remarkable was not that it stored energy, but that it **flowed**. It was used, regenerated, and used again, constantly. ATP was not a static reserve. It was a dynamic process unfolding in time.

That single fact changes everything. Your body does not stockpile ATP the way a car stores gasoline. At any moment, ATP is being consumed and replenished at the same time. The amount available rises and falls as conditions change. When exercise begins, ATP availability drops. As metabolism responds, production increases. Sometimes ATP stabilizes or even recovers. Eventually, fatigue sets in and availability declines again. This story is not random. It has structure. It has turning points. It has shape.

Once something changes in time, it becomes a **function**. And once something is a function, it has a graph. ATP availability can be drawn as a curve — not because we like graphs, but because curves are the natural language of change. That curve has a minimum when ATP is most stressed, a maximum when recovery peaks, and a change in concavity when the body shifts strategies. Those features are not guesses. They are enforced by mathematics.

This is where calculus quietly enters biology. Calculus was invented to understand motion, but motion is not only about position and velocity. It is about **rates**. How fast something changes, and how that rate itself changes. In mechanics we track position, velocity, and acceleration. In biology we can track ATP availability, its rate of change, and how that rate is changing. Different systems — Same logic and same mathematics.

Critical points on a curve — maxima, minima, and inflection points — are not found by looking. They are found by reasoning. A maximum occurs when the first derivative changes from positive to negative. A minimum occurs when it changes from negative to positive. The second derivative tells us *why* the turn happens — whether the curve bends like a hill or a bowl. Zero slope alone is not enough. Direction change is the key. This rule applies to falling objects, racing cars, & molecules inside muscle cells alike.

There is an old story from ancient Iran about the Parthians, famous horse archers who would appear to retreat from battle, only to turn back in the saddle and release one final, devastating arrow. Historians called it the **Parthian shot**. Calculus has its own version. After all the algebra and tables, one idea remains standing: *you do not need the exact curve to understand the motion*. If you know how derivatives behave, you can reconstruct the story.

That final sketch — built from signs, slopes, and concavity — is the real lesson. ATP did not just teach us how muscles move. It shows us why calculus exists at all: to read change, predict turning points, and understand the hidden structure behind motion. That insight is the parting shot — and it carries far beyond this page.

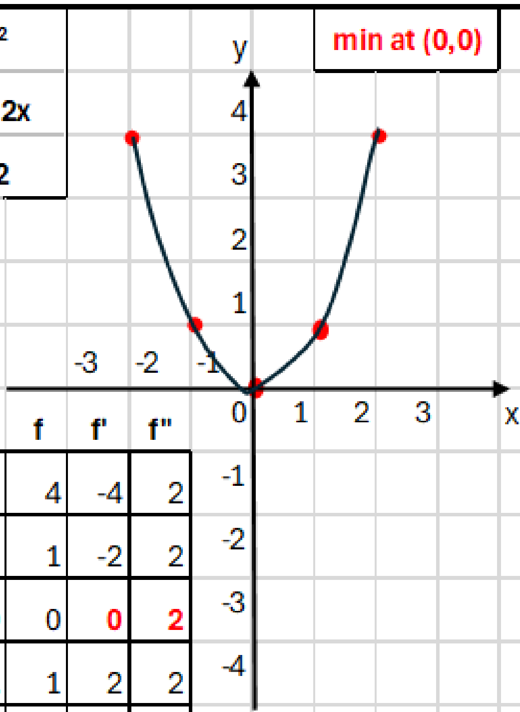
Critical Points: Maximum (Max), Minimum (Min), and Inflection Points

f(x) = x²

f'(x) = 2x

f'(x) = 2

min at (0,0)



x	f	f'	f''
-2	4	-4	2
-1	1	-2	2
0	0	0	2
1	1	2	2
2	4	4	2

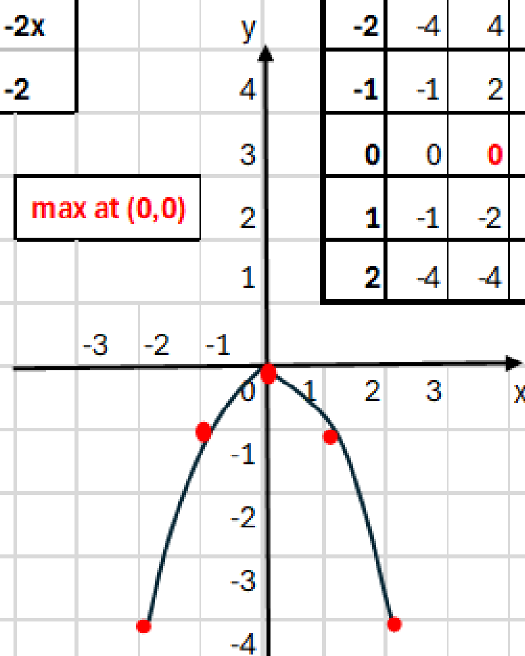
f(x) =- x²

f'(x) = -2x

f''(x) = -2

x	f	f'	f''
-2	-4	4	-2
-1	-1	2	-2
0	0	0	-2
1	-1	-2	-2
2	-4	-4	-2

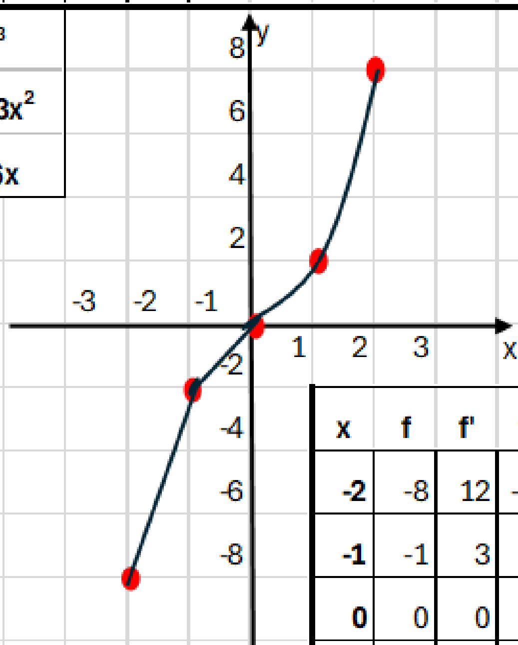
max at (0,0)



f(x) = x³

f'(x) = 3x²

f''(x) =6x

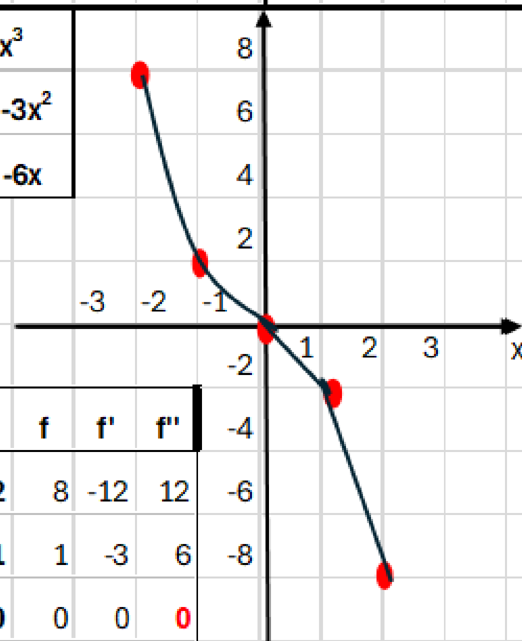


x	f	f'	f''
-2	-8	12	-12
-1	-1	3	-6
0	0	0	0
1	1	3	6
2	8	12	12

f(x) = -x³

f'(x) = -3x²

f''(x) = -6x



x	f	f'	f''
-2	8	-12	12
-1	1	-3	6
0	0	0	0
1	-1	-3	-6
2	-8	-12	-12

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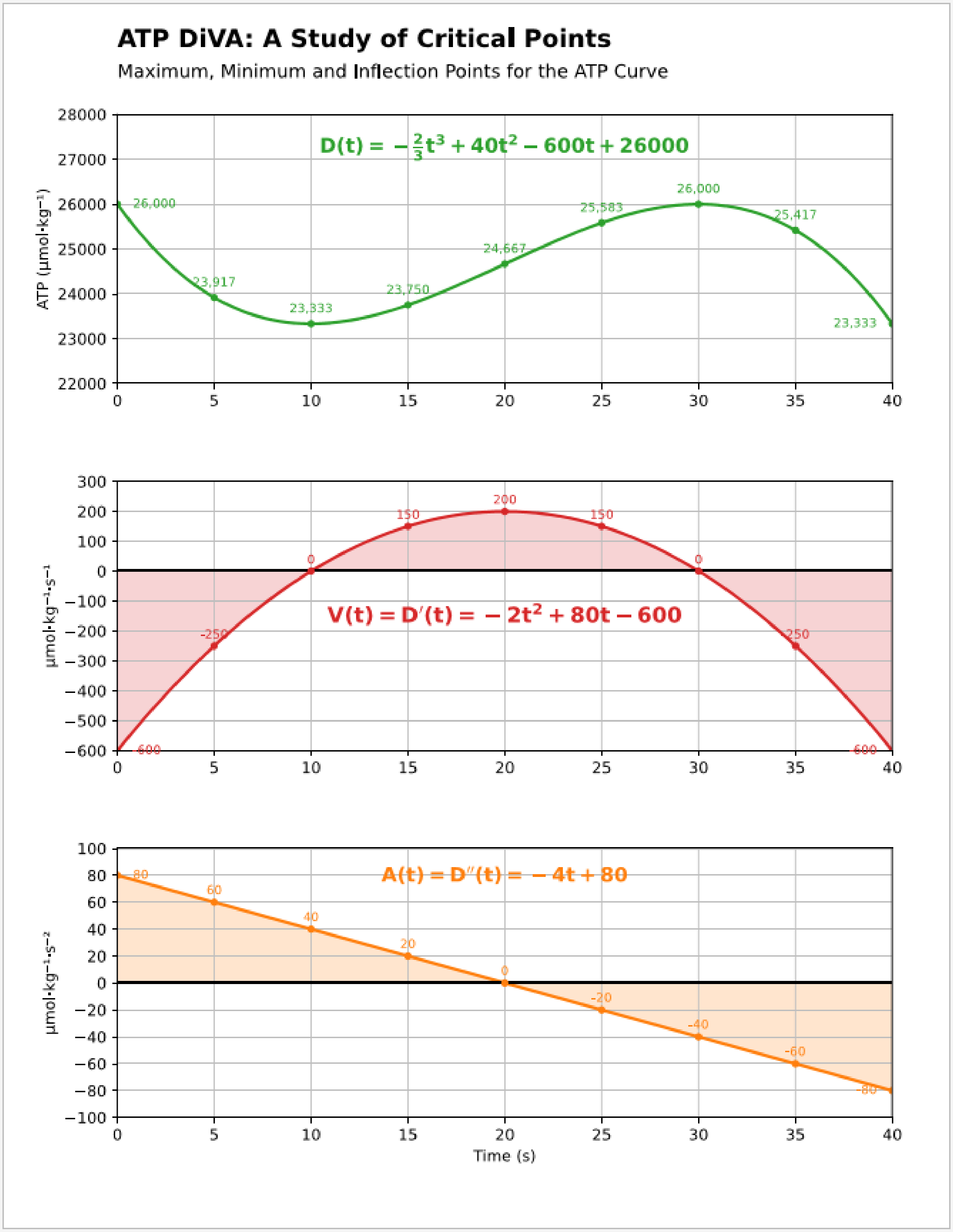
📘 Adventure 13– Student Worksheets and Chart Solutions – Activity 2



Adventure 13 – Student Activities 1 and 2 Solutions

ATP DiVA: A Study of Critical Points

Maximum, Minimum, and Inflection Points for the ATP Curve

Review of Critical Points: Maximum (Max), Minimum (Min), and Inflection Points

Complete the tables and draw the curves for $f(x)=x^2$, $f(x)=-x^2$, $f(x)=x^3$, and $f(x)=-x^3$ and complete the following sentences:

For $f(x)=x^2$ we have a **minimum (min)** at the point (0,0) because f'(0**) = 0** and f''(0) < 0

For f(x)=-x2 we have a **maximum (max)** at the point (0,0) because f'(0) = **0** and f''(0) **>0**

For $f(x)=x^3$ we have an **inflection point** at (0,0) because f''(0) = **0**

For $f(x)=-x^3$ we have an **inflection point** at (0,0) because f''(0)= **0**

Purpose

In the remainder of this activity, you will **build a curve from calculus**, not just from plotting points.

You will:

- start with an equation for ATP availability,
- compute derivatives,
- use values and signs to understand behavior,
- and sketch the ATP Availability curve using **critical points**.

You will **not** be given the curve at first.

Given: ATP Availability Model

$$\boldsymbol{D(t) = -\frac{2}{3}t^3 + 40t^2 - 600\text{t} + 26,000}$$

Where:

- t= time (seconds)
- $D(t)$= ATP availability (μmol·kg^{-1} dry muscle) (micromole per kilogram)
    - Scientists use a mole to count **6.022 × 10²³ molecules at a time**, because molecules are incredibly small.
    - **1 mole (mol)** means **6.022 × 10²³** particles — this number is called **Avogadro's number**.
    - **1 mole (mol)** = base unit

- **1 millimole (mmol)** = 10^{-3}moles = **0.001 mol**
- **1 micromole (µmol)** = 10^{-6}moles = **0.000001 mol**

You are given a **blank DiVA chart** with:

- axes labeled
- units shown
- gridlines
- **no curves drawn**

Part 1 — Find the Derivatives

1. First derivative (rate of ATP change)

$$V(t) = D'(t)$$

Write your result: $\boxed{\boldsymbol{V(t) = -2t^2 + 80t - 600}}$

2. Second derivative (change of the rate)

$$A(t) = D''(t)$$

Write your result: $\boxed{\boldsymbol{A(t) = -4\text{t} + 80}}$

Part 2 — Draw the Acceleration Curve $\boldsymbol{A(t)}$

On the **bottom graph** of the DiVA:

- Find A(0) and mark it on the chart $\boxed{\boldsymbol{D(0) = 80}}$
- Find for what value of t A(t)= 0 $\boxed{\boldsymbol{A(t) = 0}}$ ➔ $\boxed{\boldsymbol{0 = -4\text{t} + 80}}$ ➔ $\boxed{\boldsymbol{t = 20}}$
- Draw $A(t)$
- Find the slope of A(t) $\boxed{\boldsymbol{Slope\ of\ A(t) = -4}}$
- Find the Intercept of A(t) $\boxed{\boldsymbol{Intercept\ of\ A(t) = 80}}$

Question: What is the slope of V(t) when A(t) = 0? $\boxed{\boldsymbol{Slope\ of\ V(t) is\ 0}}$

Part 3 — Draw the Velocity Curve $\boxed{\boldsymbol{V(t) = -2t^2 + 80t - 600}}$

- Find the root of V(t) (where it crosses the t-axis)
  Remember
  $\boxed{\boldsymbol{V(t) = 0 = (\mathrm{V - V1})(\mathrm{V - V2}) = t^2 - 2mt + p = t^2 - 40t + 600}}$

Then $V_1+V_2=-2m$ and $V_1V_2=p$ and V_1 and V_2 are where V(t) crosses the t axis

$V_1+V_2=40$ and $V_1V_2=300$

$V_1 = 10$ and .$V_2= 30$

Also, you could find the roots with the following formula:

$$V_1 = m+\sqrt{(m^2-p)} \quad \text{and } V_2 = m-\sqrt{(m^2-p)}$$

We get the same roots this way: **Yes** **NO**

On the **middle graph** of the DiVA:

- Use the fact that $A(t)$ **is the slope of** $V(t)$
- Sketch $V(t)$
- Clearly mark where: $V(t) = 0$
- These will become the **critical points of** $\boldsymbol{D(t)}$.

Part 4 —Table for "drawing the curve using critical points 10, 20 and 30 Seconds"

For the critical points t=10 and t=30 specify the value for V and for Critical point t= 20 the value for A

Table 1. Sign Table for Drawing D (ATP Availability)

Interval (seconds)	0–10	10	10–20	20	20–30	30	30–40
D (ATP availability)	−	min	+	Inflection	+	max	−
V = D'	−	0	+	+	+	0	−
A = V' = D"	+	+	+	0	−	−	−

$$\boldsymbol{V(t) = -2t^2 + 80t - 600} \qquad \boldsymbol{A(t) = -4t + 80}$$

Using the formulas for V and D completes the values in the following table.

Table 2 . Values for V(t) and D(t)

t	0	5	10	15	20	25	30	35	40
V(t)	-600	250	0	150	200	150	0	250	-600
A(t)	80	60	40	20	0	-20	-40	-60	-80

Use the values for in Table 2 to put signs in Table 1.

Find and mark the maximum and minimum and the inflection point for D(t) in Table 1 using the following

First derivative decides *if* there is an extremum (max or min). Second derivative decides *which kind*.

In Table 1 at t=10 we have a minimum since V(10)=0 and A(10) = 40>0

In Table 1 at t=30 we have a maximum since V(30)=0 and A(30)= -40<0

In Table 1 at t=20 we have an inflection point since A(30)=0

Part 5 — Draw the Curve for D(t) using the sign table and the Value Table

Table 3 . Values for D(t))

t	0	10	20	30	40
D(t))	26000	23,333	24,666	26,000	23,333

$$D(t) = -\frac{2}{3}t^3 + 40t^2 - 600\mathrm{t} + 26,000$$

D(0) = 26,000

D(10) = 23,333

D(20) = 24,666

D(30) = 26,000

D(40) = 23,333

Part 5 — Sketch the ATP Availability Curve

On the **top graph** of the DiVA:

- Mark where $V(t) = 0$ (min/max)
- Mark where $A(t) = 0$ (inflection)
- Use Table 1 and Table 3 to draw a **smooth, reasoned sketch**.

Part 6. Compare your Charts with the DiVA Charts produced by ChatGPT

My Chart are very similar to the DiVA Charts Produced by ChatGPT **Yes** **No**

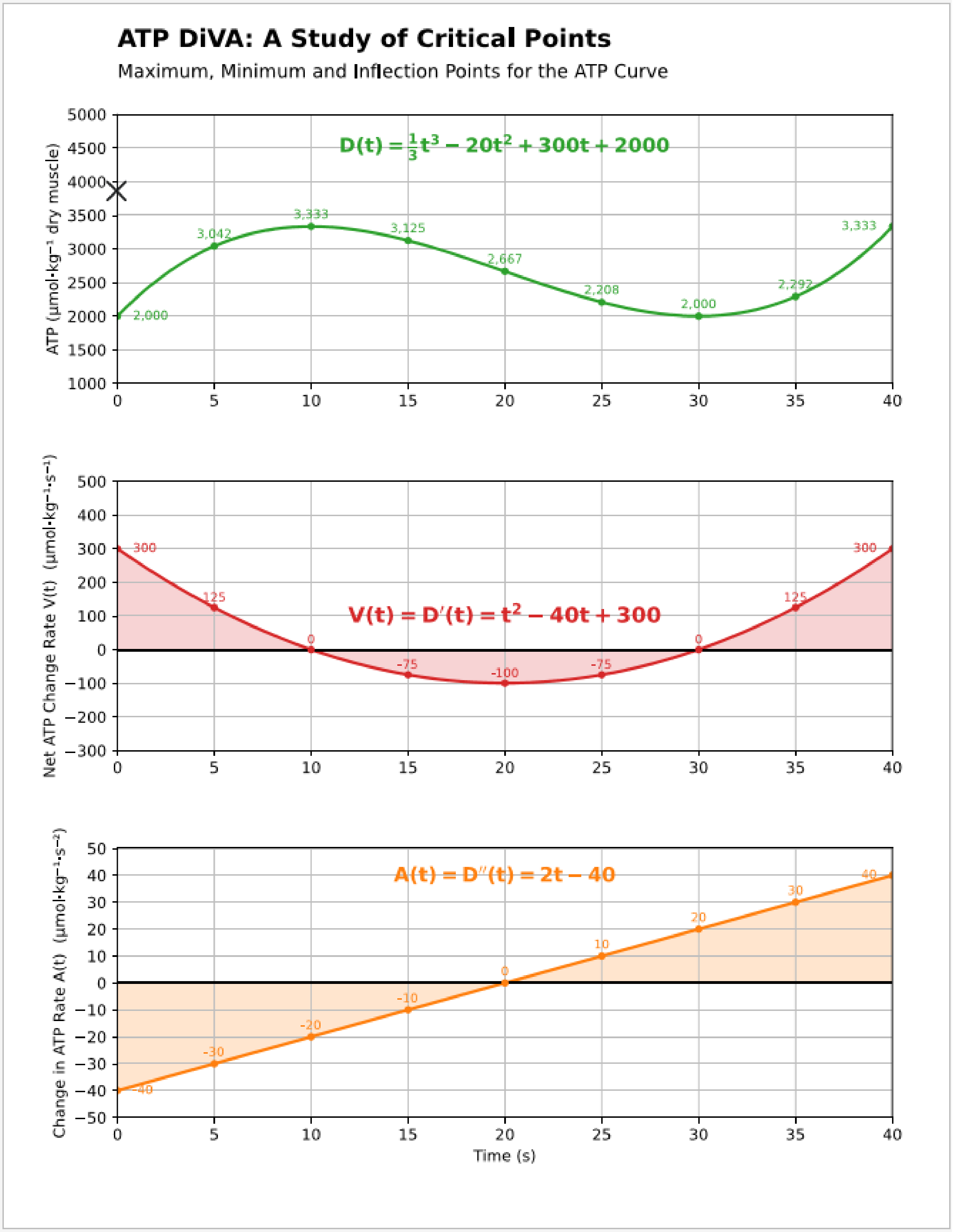
Explain What is Happening with ATP Availability:

Part 7 — The Parting Shot

Without recalculating any values:

1. Explain how you could sketch the shape of ATP curve using **only**
   - zeros of $V(t)$,
   - zeros of $A(t)$,
   - and signs.
2. Why is this a very powerful idea in calculus? Write a sentence or two:

   This means that we can completely see the shape of a function by finding it's critical points.



ATP DiVA: A Study of Critical Points

Maximum, Minimum, and Inflection Points for the ATP Curve

Review of Critical Points: Maximum (Max), Minimum (Min), and Inflection Points

Purpose

In this activity you will **build a curve from calculus**, not just from plotting points.

This ATP model represents an athlete who **boosts ATP availability early**—for example, by consuming energy drinks or stimulants—followed later by a **second boost when fatigue sets in**.

Given: ATP Availability Model: $\boldsymbol{D(t) = \frac{t^3}{3} - 20t^2 + 300t + 2000}$

Where: t= time (seconds), $D(t)$= ATP availability (μmol·kg^{-1} dry muscle) (micromole per kilogram): Scientists use a mole to count **6.022 × 10^{23} molecules at a time**, **1 micromole (μmol)** = 10^{-6}moles = **0.000001 mol**

Part 1 — Find the Derivatives

$$\boldsymbol{D(t) = \frac{t^3}{3} - 20t^2 + 300t + 2000}$$

1. First derivative (rate of ATP change)

$V(t) = D'(t)$ Write your result: $\boldsymbol{V(t) = t^2 - 40t + 300}$

2. Second derivative (change of the rate)

$A(t) = D''(t)$ Write your result: $\boldsymbol{A(t) = 2t - 40}$

Part 2 — Draw the Acceleration Curve $\boldsymbol{A(t)}$

On the **bottom graph** of the DiVA:

- Find A(0) and mark it on the chart $\boldsymbol{A(0) = -40}$
- For what value of t, A(t)= 0 $\boldsymbol{A(t) = 0}$ → $\boldsymbol{2t - 40 = 0}$ → $\boldsymbol{t = 20}$
- **Draw $\boldsymbol{A(t)}$**
- Find the slope of A(t) $\boldsymbol{Slope\ of\ A(t) = 2}$
- Find the Intercept of A(t) $\boldsymbol{Intercept\ of\ A(t) = -40}$

Question: What is the slope of V(t) when A(t) = 0? $\boldsymbol{Slope\ of\ V(t) is\ 0}$

$$\boxed{V(t) = t^2 - 40t + 300}$$

- Find the root of V(t) (where it crosses the t-axis)

$$\boxed{V(t) = 0 = (V - V1)(V - V2) = t^2 - 2mt + p} \quad \boxed{t^2 - 40\,t + 300 \quad = 0}$$

Then V_1+V_2=2m and V_1V_2=p and V_1 and V_2 are where V(t) crosses the t axis

V_1+V_2= 40 and V_1V_2= 300

V_1 = 10 and .V_2= 30

Also, you could find the roots with the following formula:

m= 20 , √(m²-p) = √400-300 = √100 = 10

V_1 =m+√(m²-p) = 20+10 = 30 V_2 = m- √(m²-p)=20 – 10 = 10

Do you get the same roots this way?: <u>**Yes**</u> **NO**

On the **middle graph** of the DiVA:

- Find and mark V(0) = 300
- Find V(40) = 1,600-40x40+300= 300
- Clearly mark where: $V(t) = 0$: For t = 10 and t = 30 V(t) = 0
- Find and mark V(20) = 400-800+300= -100 this is where the slope of V(t) is 0
- **Sketch $V(t)$ on the DiVA middle area.**
- **The points t =10, 20 and 30 are critical points of $D(t)$.**

Part 4 —Drawing the D(t) curve using critical points 10, 20 and 30 Seconds"

$$\boxed{V(t) = t^2 - 40t + 300} \quad \boxed{A(t) = 2t - 40}$$

Using the formulas for V(t) and A(t) complete Table 1:

Table 1 . Values for V(t) and A(t)

t	0	5	10	15	20	25	30	35	40
V(t)	300	125	0	-75	-100	-75	0	125	300
A(t)	-40	-30	-20	-10	0	10	20	30	40

Table 2 . Values for D(t) for t=0 ,40 and at Critical Points

t	0	10	20	30	40
D(t))	2,000	+3,333	2,667	2,000	3,333

Then use the values for in **Table 2** to put the values for critical points for **D(t)**. Finally, use the values from **Table 1** to put in zeros for the critical points for **V(t)** and **A(t)** and signs in all the ranges for **V(t)** and **A(t)**.

Table 3. Critical Values and Sign Table for Drawing D (ATP Availability)

Interval (seconds)	0	0-10	10	10–20	20	20–30	30	30–40	40
D (ATP Availability)	2,000		3,333 max		2,667 Inflection		2,000 min		3,333
V = D'	+	+	0	-	-	-	0	+	+
A=V'	-	-	-	-	0	+	+	+	+

Find and mark the max & min and the inflection point for D(t) in Table 3 using these facts: **First derivative decides *if* there is an extremum (max or min). Second derivative decides *which kind of extremum it is:***

In Table 2 at t=10 we have a: Maximum (max) because A(10)<0

In Table 2 at t=30 we have a Minimum (min) because A(30)>0

In Table 2 at t=20 we have an: Inflection Point because A(t)=0

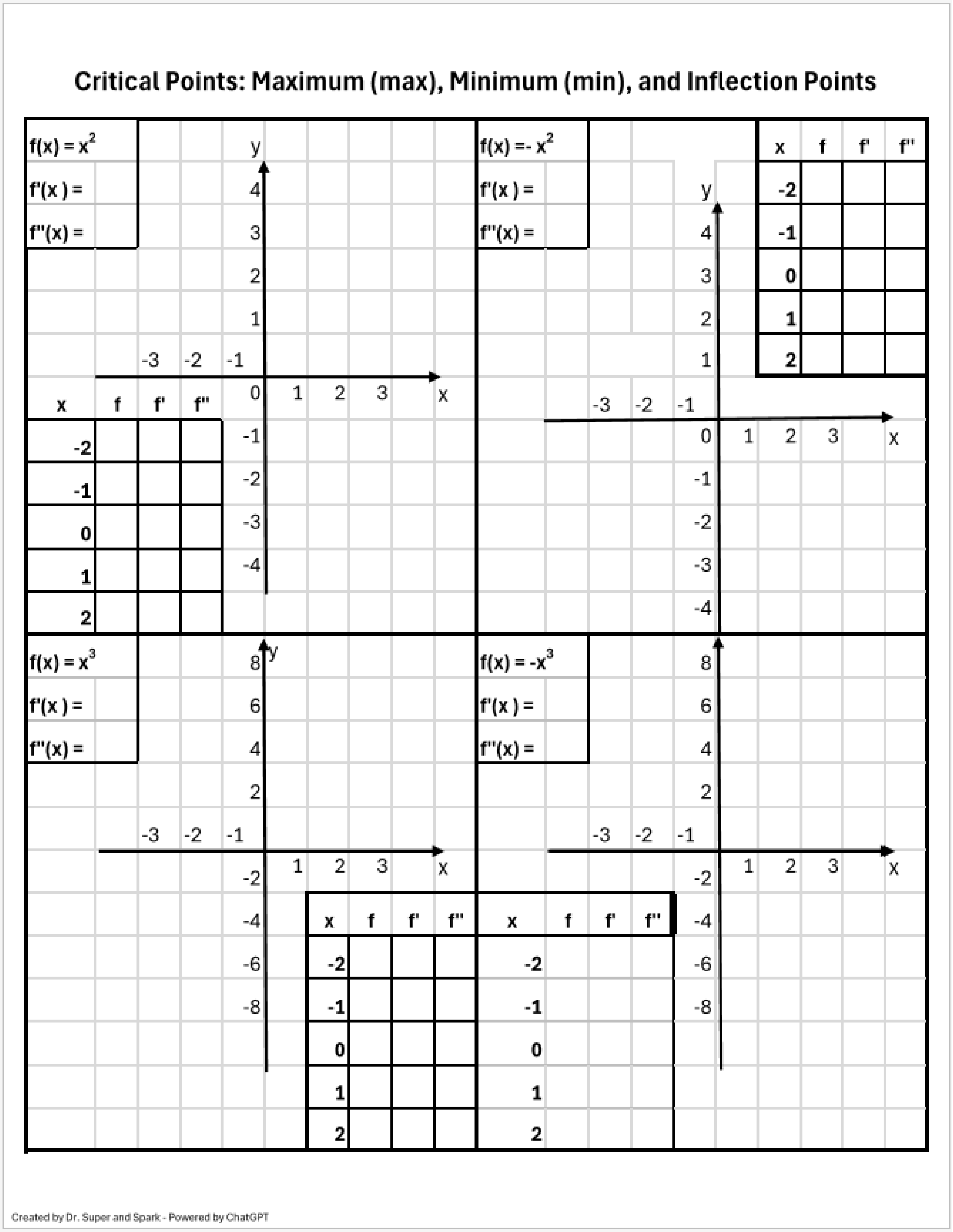
Now draw the Curve for D(t) using Table 3.

Part 5. Compare your Charts with the DiVA Charts produced by ChatGPT

My Chart are very similar to the DiVA Charts Produced by ChatGPT   Yes   No

Explain What is Happening with ATP Availability based on D(t): The person doing the activity took some energy producing food or substance then exercised. The ATP reserves reached a maximum nd then started dropping going thru and inflection point and reaching a minimum where he took some more energy boosters.

📘 Adventure 13– Student Worksheets and Chart – Activity 1 and 2



x	f	f'	f''
-2			
-1			
0			
1			
2			

x	f	f'	f''
-2			
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x	f	f'	f''
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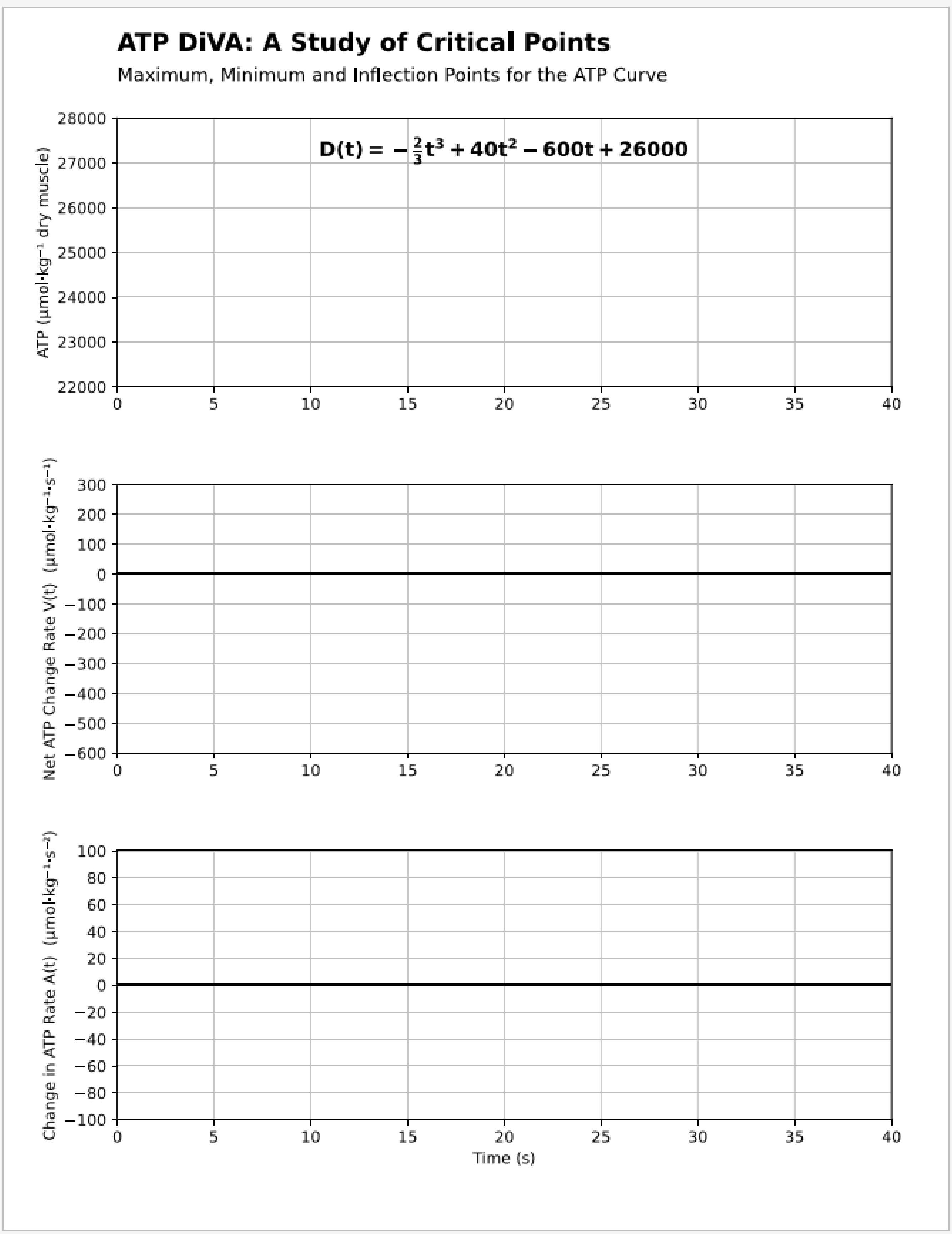

ATP DiVA: A Study of Critical Points
Maximum, Minimum and Inflection Points for the ATP Curve
$D(t) = -\frac{2}{3}t^3 + 40t^2 - 600t + 26000$
ATP (μmol•kg⁻¹ dry muscle)
Net ATP Change Rate V(t) (μmol•kg⁻¹•s⁻¹)
Change in ATP Rate A(t) (μmol•kg⁻¹•s⁻²)
Time (s)

📘 Adventure 13 – Student Activity worksheets 1 and 2

ATP DiVA: A Study of Critical Points

Maximum, Minimum, and Inflection Points for the ATP Curve

Review of Critical Points: Maximum (Max), Minimum (Min), and Inflection Points

Complete the tables and draw the curves for $f(x)=x^2$, $f(x)=-x^2$, $f(x)=x^3$, and $f(x)=-x^3$ and complete the following sentences:

For $f(x)=x^2$ we have a ________________ at the point (0,0) because f'(0**) =** ___ and f''(0) ___

For $f(x)=-x^2$ we have a _______________ at the point (0,0) because f'(0) = ___ and f''(0) ___

For $f(x)=x^3$ we have an ________________ at (0,0) because f''0) = _______

For $f(x)=-x^3$ we have an ______________________at (0,0) because f''(0) = _____

Purpose

In the remainder of this activity you will **build a curve from calculus**, not just from plotting points.

You will:

- start with an equation for ATP availability,
- compute derivatives,
- use values and signs to understand behavior,
- and sketch the ATP Availability curve using **critical points**.

You will **not** be given the curve at first.

Given: ATP Availability Model

$$\boldsymbol{D(t) = -\frac{2}{3}t^3 + 40t^2 - 600t + 26{,}000}$$

Where:

- t= time (seconds)
- $D(t)$= ATP availability (μmol·kg^{-1} dry muscle) (micromole per kilogram)
  - Scientists use a mole to count **6.022 × 10²³ molecules at a time**, because molecules are incredibly small.
  - **1 mole (mol)** means **6.022 × 10²³** particles — this number is called **Avogadro's number**.
  - **1 mole (mol)** = base unit

- **1 millimole (mmol)** = 10^{-3}moles = **0.001 mol**
- **1 micromole (µmol)** = 10^{-6}moles = **0.000001 mol**

You are given a **blank DiVA chart** with: axes labeled, units shown, gridlines, **no curves drawn**

Part 1 — Find the Derivatives

$$\boldsymbol{D(t) = -\frac{2}{3}t^3 + 40t^2 - 600\mathrm{t} + 26,000}$$

1. First derivative (rate of ATP change)

$$V(t) = D'(t)$$

Write your result: $\boldsymbol{V(t) =}$

2. Second derivative (change of the rate)

$$A(t) = D''(t)$$

Write your result: $\boldsymbol{A(t) =}$

Part 2 — Draw the Acceleration Curve $\boldsymbol{A(t)}$

On the **bottom graph** of the DiVA:

- Find A(0) and mark it on the chart $\boldsymbol{D(0) =}$
- Find for what value of t, A(t)= 0

  $\boldsymbol{A(t) = 0}$ ➔ $\boldsymbol{\qquad\qquad = 0}$ ➔ $\boldsymbol{t =}$
- **Draw** $\boldsymbol{A(t)}$
- Find the slope of A(t) $\boldsymbol{Slope\ of\ A(t) =}$
- Find the Intercept of A(t) $\boldsymbol{Intercept\ of\, A(t) =}$

Question: What is the slope of V(t) when A(t) = 0? $\boldsymbol{Slope\ of\ V(t) is}$

Part 3 — Draw the Velocity Curve

$$\boldsymbol{V(t) = -2t^2 + 80t - 600}$$

- Find the root of V(t) (where it crosses the t-axis)

Remember that you can divide the equation $-2t^2+80t-600=0$ by the coefficient of t^2 that is -2.

$$\boxed{V(t) = 0 = (V - V1)(V - V2) = t^2 - 2mt + p} \quad \boxed{t^2 - \quad t - \quad = 0}$$

Then $V_1+V_2=2m$ and $V_1V_2=p$ and V_1 and V_2 are where V(t) crosses the t axis

$V_1+V_2=$_____________ and $V_1V_2=$ ______________

$V_1 =$ ______ and .$V_2=$ _____

Also, you could find the roots with the following formula:

m=_____, $\sqrt{(m^2-p)}$ = ______________________

$V_1 = m+\sqrt{(m^2-p)}$ = ___________ $V_2 = m - \sqrt{(m^2-p)}$=______________

Do you get the same roots this way?: **Yes NO**

On the **middle graph** of the DiVA:

- Find and mark V(0) = _________________
- Find V(40) = ___________________
- Clearly mark where: $V(t) = 0$ for t = ______ and t = ______ V(t) = 0
- Find and mark V(20) = ______________ this is where the slope of V(t) is 0
- **Sketch $V(t)$ on the DiVA middle area.**
- **The points t =10, 20 and 30 are critical points of $D(t)$.**

Part 4 —Drawing the D(t) curve using critical points 10, 20 and 30 Seconds”

$V(t) =$ ______ $A(t) =$ ______

Using the formulas for V and D complete following table.

Table 1 . Values for V(t) and D(t)

t	0	5	10	15	20	25	30	35	40
V(t)									
A(t)									

Use the values for in Table 1 to put signs in Table 2.

For the critical points t=10 and t=30 specify the value for V and for critical point t= 20 the value for A.

Then use the values for in Table 2 to put signs in Table 1.

Table 2. Sign Table for Drawing D (ATP Availability)

Interval (seconds)	0–10	10	10–20	20	20–30	30	30–40
D (ATP availability)		______		______	______		
V = D'							
A=V’							

Find and mark the maximum and minimum and the inflection point for D(t) in Table 3 using the following facts:

First derivative decides *if* there is an extremum (max or min). Second derivative decides *which kind*.

In Table 2 at t=10 we have a ____

In Table 2 at t=30 we have a ____

In Table 2 at t=20 we have an ______________________________

Now draw the Curve for D(t) using the sign table and the following table.

Table 3 . Values for D(t) at Critical Points

t	0	10	20	30	40
D(t))	26,000	23, 333	24,667	26,000	23,333

Part 5. Compare your Charts with the DiVA Charts produced by ChatGPT

My Chart are very similar to the DiVA Charts Produced by ChatGPT Yes No

Explain What is Happening with ATP Availability:

___

Part 6 — The Parting Shot

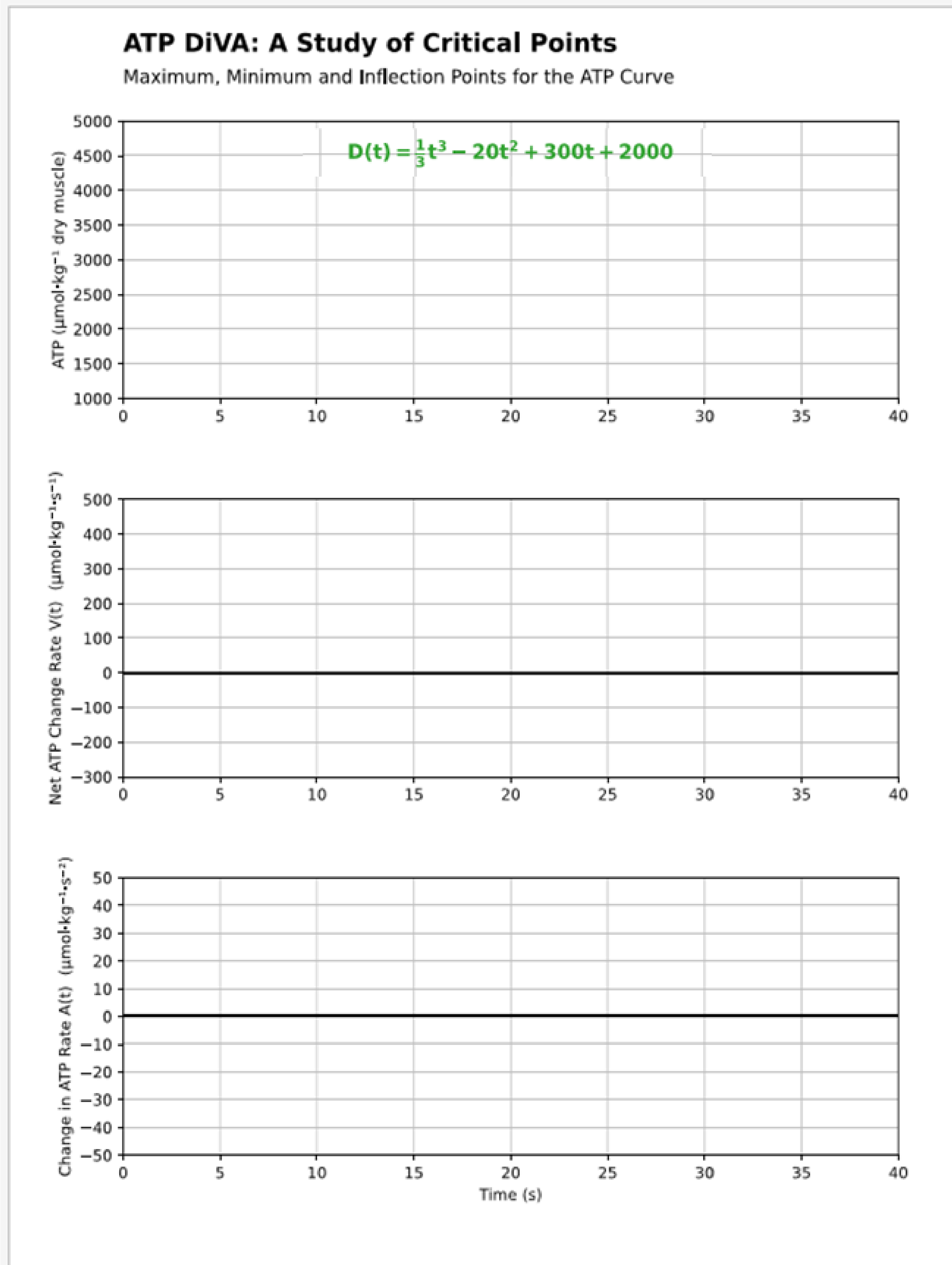
Without recalculating any values:

3. Explain how you could sketch the shape of ATP curve using **only**
    - zeros of $V(t)$,
    - zeros of $A(t)$,
    - and signs.
    - and just the values for the critical points
4. Why is this a very powerful idea in calculus? Write a sentence or two:

___

___

📘 Adventure 13 Student Activity 3 Chart and Worksheets



📘 Adventure 13 Student Activity 3 worksheets

ATP DiVA: A Study of Critical Points

Maximum, Minimum, and Inflection Points for the ATP Curve

Review of Critical Points: Maximum (Max), Minimum (Min), and Inflection Points

Purpose

In this activity you will **build a curve from calculus**, not just from plotting points.

This ATP model represents an athlete who **boosts ATP availability early**—for example, by consuming energy drinks or stimulants—followed later by a **second boost when fatigue sets in**.

Given: ATP Availability Model: $\boxed{D(t) = \frac{t^3}{3} - 20t^2 + 300t + 2000}$

Where: t= time (seconds), $D(t)$= ATP availability (μmol·kg^{-1} dry muscle) (micromole per kilogram): Scientists use a mole to count **6.022 × 10^{23} molecules at a time**, **1 micromole (μmol)** = 10^{-6}moles = **0.000001 mol**

Part 1 — Find the Derivatives

$$\boxed{D(t) = \frac{t^3}{3} - 20t^2 + 300t + 2000}$$

1. First derivative (rate of ATP change)

$V(t) = D'(t)$ Write your result: $\boxed{V(t) = \qquad}$

2. Second derivative (change of the rate)

$A(t) = D''(t)$ Write your result: $\boxed{A(t) = \qquad}$

Part 2 — Draw the Acceleration Curve $A(t)$

On the **bottom graph** of the DiVA:

- Find A(0) and mark it on the chart $\boxed{A(0) = \qquad}$
- For what value of t, A(t)= 0 $\boxed{A(t) = 0}$ ➔ $\boxed{\qquad\qquad}$ ➔ $\boxed{t = \qquad}$
- **Draw $A(t)$**
- Find the slope of A(t) $\boxed{Slope\ of\ A(t) = \qquad}$
- Find the Intercept of A(t) $\boxed{Intercept\ of\ A(t) = \qquad}$

Question: What is the slope of V(t) when A(t) = 0? $\boxed{\textit{Slope of } V(t) \textit{is} \quad\quad\quad}$

Part 3 — Draw the Velocity Curve

$$\boxed{V(t) = t^2 - 40t + 300}$$

- Find the root of V(t) (where it crosses the t-axis)

$$\boxed{V(t) = 0 = (V - V1)(V - V2) = t^2 - 2mt + p} \quad \boxed{\quad\quad\quad\quad\quad\quad = 0}$$

Then $V_1 + V_2 = 2m$ and $V_1 V_2 = p$ and V_1 and V_2 are where V(t) crosses the t axis

$V_1 + V_2 =$ ______ and $V_1 V_2 =$ _________

$V_1 =$ ___ and .$V_2 =$ ____

Also, you could find the roots with the following formula:

m= ___ , √(m²-p) = ____________________________

V_1 =m+√(m²-p) = __________ V_2 = m- √(m²-p)= ______________

Do you get the same roots this way?: **Yes NO**

On the **middle graph** of the DiVA:

- Find and mark V(0) = _______
- Find V(40) = ___________________
- Clearly mark where: $V(t) = 0$: For t = ____ and t = ___ V(t) = 0
- Find and mark V(20) = _____________________ this is where the slope of V(t) is 0
- **Sketch $V(t)$ on the DiVA middle area.**
- **The points t =10, 20 and 30 are critical points of $D(t)$.**

Part 4 —Drawing the D(t) curve using critical points 10, 20 and 30 Seconds”

$$\boxed{V(t) = t^2 - 40t + 300} \quad \boxed{A(t) = 2t - 40}$$

Using the formulas for V(t) and A(t) complete Table 1:

Table 1 . Values for V(t) and D(t)

t	0	5	10	15	20	25	30	35	40
V(t)									
A(t)									

Table 2 . Values for D(t) fo t=0 ,40 and at Critical Points

t	0	10	20	30	40
D(t))	2,000	+3,333	2,667	2,000	3,333

Then use the values for in **Table 2** to put the values for critical points for **D(t)**. Finally, use the values from **Table 1** to put in zeros for the critical points for **V(t)** and **A(t)** and signs in all the ranges for **V(t)** and **A(t)**.

Table 3. Critical Values and Sign Table for Drawing D (ATP Availability)

Interval (seconds)	0	0-10	10	10–20	20	20–30	30	30–40	40
D (ATP Availability)									
V = D'									
A=V’									

Find and mark the max & min and the inflection point for D(t) in Table 3 using these facts:

First derivative decides *if* there is an extremum (max or min). Second derivative decides *which kind of extremum it is:*

In Table 2 at t=10 we have a: ______________________________

In Table 2 at t=30 we have a _______________________________

In Table 2 at t=20 we have an: _____________________________

Now draw the Curve for D(t) using Table 3.

Part 5. Compare your Charts with the DiVA Charts produced by ChatGPT

My Chart are very similar to the DiVA Charts Produced by ChatGPT Yes No

Explain What is Happening with ATP Availability based on D(t):

🧑‍🏫 Adventure 14 – The Mathematics behind how Moon Circles the Earth

🎯 Purpose of Adventure 14

Adventure 14 introduces students to one of the most beautiful applications of mathematics and physics:

How can gravity keep the Moon moving in a circle around Earth?

Students discover that:

- gravity acts continuously
- acceleration can change direction without changing speed
- circular motion requires a constant inward acceleration
- Newton's law of gravitation predicts the Moon's orbital speed
- mathematics can accurately predict the lunar month

This section connects calculus, mechanics, geometry, and astronomy into a single real-world application.

🎥 Watch First (10 Minutes)

Watch:

Why Do Magnets Work? — The One Question Feynman Refused to Explain

The purpose is not magnetism itself.

The video naturally leads students toward a deeper question:

What does science actually explain?

Ask students:

- Why do scientists sometimes say "we don't know why"?
- What is the difference between describing nature and explaining nature?
- Are forces real objects or mathematical descriptions?

Many students are surprised that science often describes *how* nature behaves without fully explaining the ultimate *why*.

📜 Read or Listen Next

Story:

The Moon That Never Falls

The story introduces:

- Newton's idea of gravity
- motion in space
- the puzzle of the Moon's orbit
- how mathematics predicts celestial motion

Encourage students to think about:

- Why the Moon does not fall to Earth
- Why it does not fly away
- What keeps it moving in a circle

These questions become the motivation for the activity.

🧭 What Students Will Do

Students will:

- Review Newton's Law of Gravitation
- Review centripetal acceleration
- Set gravitational force equal to centripetal force
- Derive the Moon's orbital speed
- Compute the Moon's orbital period
- Compare prediction with observation
- Interpret the Moon as "falling around Earth"

No advanced calculus is required.

The focus is on physical intuition and mathematical modeling.

📊 The Three Core Ideas

Page 1 — Gravity as a Force

Students begin with Newton's Law:

$$\boldsymbol{F = \frac{GM_EM_M}{r^2}}$$

Emphasize:

- stronger masses create stronger attraction
- greater distance weakens gravity
- gravity acts across empty space

Students often find this idea surprising.

Page 2 — Circular Motion Requires Acceleration

This is the most conceptually difficult section.

Students learn:

- acceleration does not always mean speeding up
- acceleration can mean changing direction
- circular motion requires continual inward acceleration

Teacher emphasis:

"The Moon is accelerating every second even though its speed remains nearly constant."

This idea often takes time to absorb.

Page 3 — Mathematics Predicts Reality

Students set

$$\frac{GM_EM_M}{r^2} = \frac{M_Mv^2}{r}$$

and derive

$$v = \sqrt{\frac{GM_E}{r}}$$

Using actual astronomical data, they obtain:

- Orbital speed ≈ 1.02 km/s
- Orbital period ≈ 27.4 days

Students are usually impressed that a simple equation reproduces the observed lunar month.

⭐ The Big Idea

Make this explicit:

Gravity provides exactly the centripetal force needed for circular motion.

And therefore:

The Moon is constantly falling toward Earth, but its sideways velocity keeps missing it.

This single idea unifies:

- gravity
- force
- acceleration
- geometry
- motion

🧠 Suggested Discussion Questions

Use only a few.

- Why doesn't the Moon fall to Earth?
- Why doesn't it fly away?

- Can something accelerate without speeding up?
- What would happen if the Moon suddenly stopped moving sideways?
- What would happen if the Moon moved much faster?
- Why is it remarkable that mathematics predicts 27.4 days?

🚀 Optional Extension

If students are curious:

Introduce the idea of escape velocity.

Ask:

How fast would the Moon need to move to escape Earth's gravity completely?

This provides a natural transition toward satellites, rockets, and space travel.

🧩 Where This Fits in the Course

Adventure 14 bridges:

- Newtonian mechanics
- circular motion
- gravity
- astronomy

It prepares students for:

- planetary motion
- Kepler's Laws
- orbital mechanics
- continuous acceleration
- deeper applications of calculus in physics

The major conceptual goal is helping students understand that acceleration is the rate of change of velocity, and velocity can change direction even when its magnitude remains constant.

Adventure 14 Story – The Moon and the Invisible Push

Newton asks whether one force rules both Earth and sky

Every night, the Moon moves. Not randomly. Not drifting aimlessly. But with remarkable precision.

It rises, crosses the sky, and sets. Over nearly four weeks its shape changes in a repeating pattern — crescent, half, full, and back again. This rhythm has been observed for as long as humans have watched the sky.

The question is not what the Moon does. The question is why. On Earth, motion usually ends. A thrown ball falls. A rolling stone stops. A flying arrow eventually reaches the ground. Gravity pulls everything downward. So why does the Moon not fall?

In the 1600s, Isaac Newton began to wonder whether the same force that pulls objects on Earth might also act on the Moon. If so, then the Moon must be constantly pulled toward Earth. But something else must also be happening, because the Moon does not crash into our planet.

It follows a curved path — steady, repeating, and predictable.

Later, physicist Richard Feynman sometimes joked that we might imagine invisible "angels" pushing objects to keep them moving. Of course, he did not mean real angels. He meant that nature behaves as if space itself contains rules that guide motion.

Today we call this idea a field — a way that space can influence how objects move.

The Moon's changing phases reveal something important. The Moon itself does not change shape. We simply see different portions of sunlight as it travels along its path. The repeating cycle shows that the Moon's motion is stable and precise.

If it moved faster, the pattern would change. If it moved slower, the pattern would change. Yet for thousands of years, the rhythm has remained almost the same.

Newton imagined that the Moon might be doing something extraordinary: constantly falling toward Earth while also moving sideways. Always falling. Never landing.

This idea transformed our understanding of the universe. It suggested that the same laws governing motion on Earth also govern the motion of the heavens.

But knowing that raises deeper questions. What determines the Moon's path? What determines its rhythm? What determines its motion through space?

That is the mystery of this adventure. The sky has been repeating the same pattern for centuries. Now it is our turn to understand why.

🌍 Adventure 14 -Student Activity Sheets - Solutions

🌙 How the Moon Circles the Earth — The Mathematics Behind It

1️⃣ The Big Idea

The Moon stays in orbit because:

Gravity provides exactly the centripetal force needed for circular motion.

That one sentence contains all the mathematics.

2️⃣ Newton's Law of Gravitation

According to Isaac Newton:

$$F_{gravity} = \frac{GM_E M_M}{r^2}$$

Where:

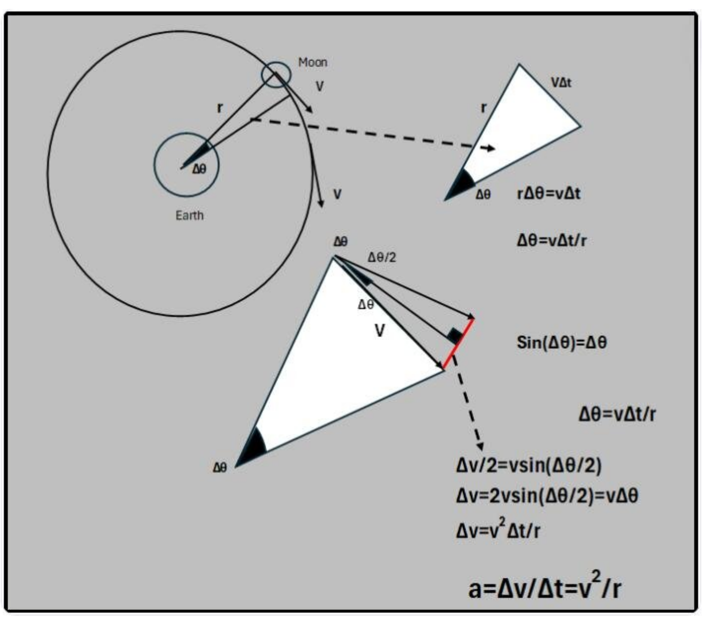
- $G = 6.67 \times 10^{-11}$
- M_E= mass of Earth
- M_M= mass of Moon
- r= distance between centers
- F= gravitational force

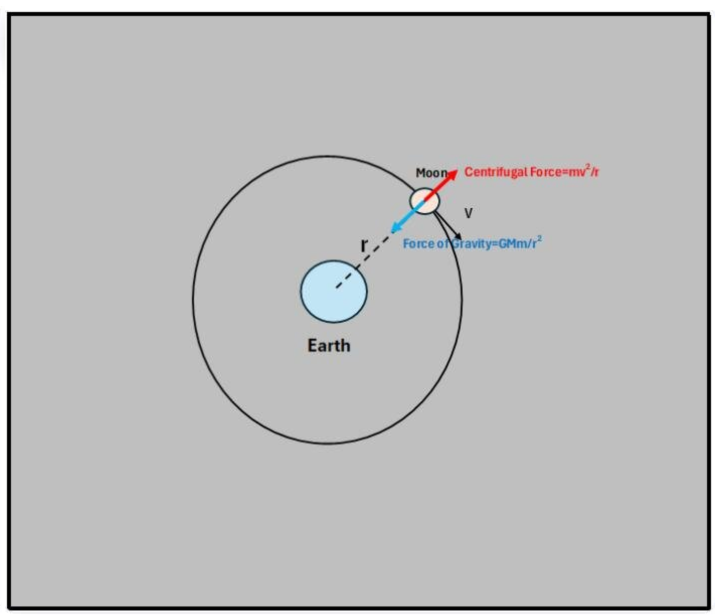
3️⃣ Circular Motion Requirement

For circular motion:

$$F_{centripetal} = \frac{M_M v^2}{r}$$

where $v$is orbital speed.





4️⃣ The Key Equation (Set Them Equal)

Since gravity is the centripetal force:

$$\frac{GM_EM_M}{r^2} = \frac{M_Mv^2}{r}$$

Cancel M_M:

$$\frac{GM_E}{r^2} = \frac{v^2}{r}$$

Multiply both sides by r:

$$\boxed{v = \sqrt{\frac{GM_E}{r}}}$$

This gives the Moon's orbital speed.

5 Plug in Real Numbers

Distance to Moon:

$$r \approx 3.84 \times 10^8 \text{ m}$$

Earth mass:

$$M_E = 5.97 \times 10^{24} \text{ kg}$$

Result:

$$v \approx 1022 \text{ m/s}$$

That's about **1 km per second**.

6 Orbital Period (Why 27.3 Days?)

The Moon travels one full circle:

$$\text{Circumference} = 2\pi r$$

Time = distance / speed:

$$T = \frac{2\pi r}{v}$$

Substitute v:

$$T = 2\pi\sqrt{\frac{r^3}{GM_E}}$$

This is a special case of **Kepler's Third Law**, discovered by Johannes Kepler.

Result:

$$T \approx 27.3 \text{ days}$$

Exactly what we observe.

7️⃣ Deeper Calculus Insight (CM-style)

Gravity creates **continuous acceleration toward Earth**:

$$a = \frac{v^2}{r}$$

But since:

$$v = \sqrt{\frac{GM_E}{r}}$$

Then:

$$a = \frac{GM_E}{r^2}$$

Which is precisely the gravitational acceleration at that distance.

So:

The Moon is constantly falling toward Earth…
but its sideways velocity keeps missing it.

That is the true mathematics of orbit.

8️⃣ Why It Doesn't Crash

If the Moon stopped moving sideways → it would fall straight down.

If it moved faster → it would escape.

In fact, if:

$$v \geq \sqrt{\frac{2GM_E}{r}}$$

🌍 Adventure 14 - Student Activity Sheets

🌙 How the Moon Circles the Earth — The Mathematics Behind It

1️⃣ The Big Idea

The Moon stays in orbit because:

Gravity provides exactly the centripetal force needed for circular motion.

That one sentence contains all the mathematics.

2️⃣ Newton's Law of Gravitation

According to Isaac Newton the gravitational Force between Earth and the Moon:

$$F_{gravity} = \frac{GM_E M_M}{r^2}$$

Where:

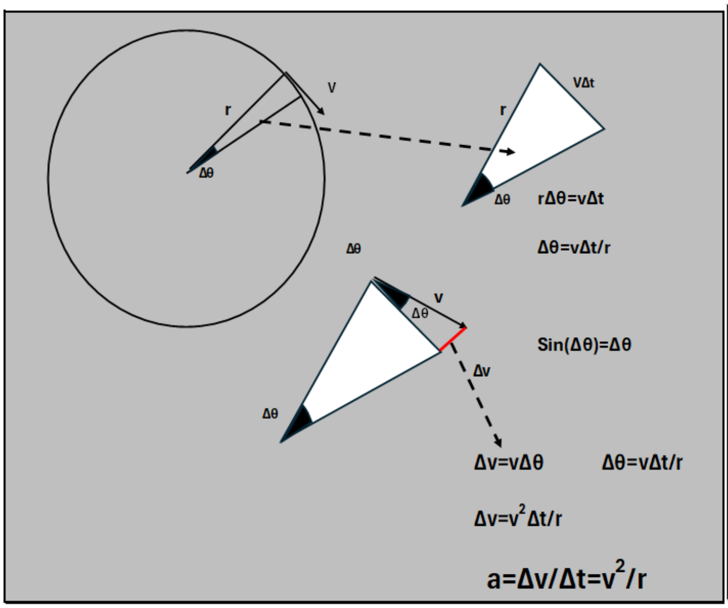
$G = 6.67 \times 10^{-11}$, M_E= mass of Earth, M_M= mass of Moon, r= distance between centers

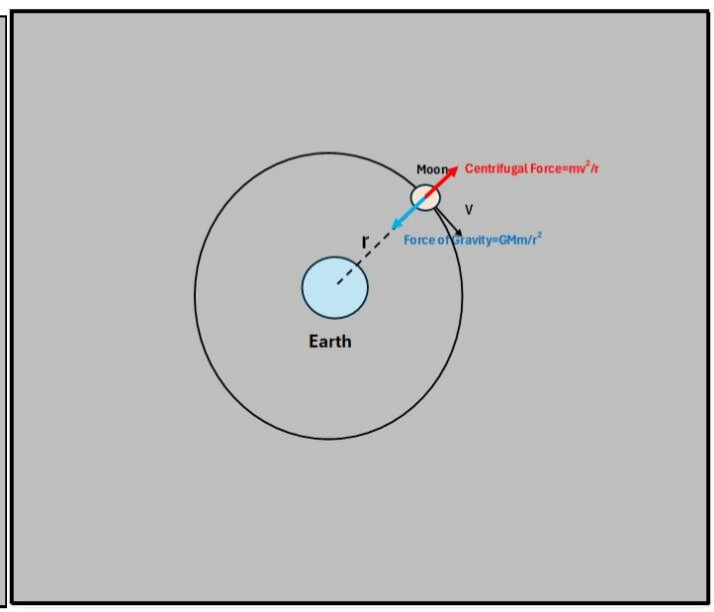
3️⃣ Circular Motion Requirement

For circular motion:

$$F_{centripetal} = \frac{M_M v^2}{r}$$

where $v$is orbital speed.





4️⃣ The Key Equation (Set Them Equal)

Since gravity and the centripetal force must cancel each other:

$$\frac{GM_E M_M}{r^2} = \frac{M_M v^2}{r}$$

Cancel M_M:

$$\frac{GM_E}{r^2} =$$

Multiply both sides by r:

$$\boxed{v = \sqrt{\frac{\quad M_E}{\quad}}}$$

This gives the Moon's orbital speed.

5 Plug in Real Numbers

Distance to Moon (center to center): $r \approx 3.84 \times 10^8$ m

Earth mass: $M_E = 5.97 \times 10^{24}$ kg

Gravitational Constant $G = 6.67 \times 10^{-11}$

Calculate V: $v \approx$ m/s

Calculate v in **km per second** $v \approx$ km/s

6 Find the Orbital Period

The Moon travels one full circle:

$$\text{Circumference} = 2\pi r$$

Time = distance / speed:

$$\boldsymbol{T = \frac{2\pi r}{v}}$$

$\mathrm{T} \approx$ second $\approx$ days

Substitute v:

$$\boldsymbol{T = 2\pi\sqrt{\frac{r^3}{GM_E}}}$$

This is a special case of **Kepler's Third Law**, discovered by Johannes Kepler.

7 Deeper Calculus Insight

Gravity creates **continuous acceleration toward Earth**:

$$a = \frac{v^2}{r}$$

But since:

$$v = \sqrt{\frac{GM_E}{r}}$$

Then:

$$a = \frac{GM_E}{r^2}$$

Which is precisely the gravitational acceleration at that distance.

So:

The Moon is constantly falling toward Earth...

but its sideways velocity keeps missing it.

That is the true mathematics of orbit.

8 Why It Doesn't Crash

If the Moon stopped moving sideways → it would fall straight down.

If it moved faster → it would escape.

In fact, if:

$$v \geq \sqrt{\frac{2GM_E}{r}}$$

it would escape completely (escape velocity).

🦹 Adventure 15 – The Sun Controls the Solar System

🧭 Overview

Adventure 15 brings together several of the central ideas developed throughout the Calculus & Mechanics series: circular motion, centripetal acceleration, Newton's law of gravitation, and planetary motion. In this activity students discover that the motion of all planets in the Solar System can be understood using a single physical constant: GM_{sun}, the gravitational parameter of the Sun.

Students begin by calculating this quantity using the known values of Newton's gravitational constant G and the mass of the Sun. Once this number is obtained, they use it to compute the orbital velocity and orbital period of several planets. By comparing these results with planetary data tables, students discover patterns in the motion of the planets and are led naturally to Kepler's Third Law.

The activity illustrates a powerful theme in physics: once a fundamental law is known, a wide range of natural phenomena can be explained with a small number of mathematical relationships.

🎯 Learning Objectives

By completing this activity students will:

• Understand that circular motion requires centripetal acceleration.
• Connect centripetal acceleration to gravitational force.
• Compute the gravitational parameter GM_{sun}.
• Calculate orbital velocity from the distance of a planet from the Sun.
• Calculate orbital periods from orbital velocity.
• Recognize patterns in planetary motion leading to Kepler's Third Law.

🧱 Background for Teachers

Previous Math Circle sessions introduced the geometric derivation of centripetal acceleration

$a = v^2 / r$

This relationship follows from analyzing the change in velocity vector as an object moves along a circular path. Even when the speed of the object remains constant, the direction of the velocity changes, producing an acceleration directed toward the center of the circle.

In planetary motion, this inward acceleration is provided by gravity. Newton's law of gravitation states

$F = GMm / r^2$

When the gravitational force acting on a planet is set equal to the centripetal force required for circular motion, the result is the orbital velocity formula

$v = \sqrt{(GM / r)}$

This formula shows that the orbital velocity of a planet depends only on its distance from the Sun and the gravitational parameter GM.

For the Sun,

$$G = 6.67 \times 10^{-11}$$

$$M_{sun} = 2 \times 10^{30}\ \text{kg}$$

Multiplying these gives

$$GM_{sun} \approx 1.33 \times 10^{20}\ \text{m}^3/\text{s}^2$$

This constant effectively determines the structure and motion of the entire Solar System.

🧮 Structure of the Activity

Students complete the following steps:

1. Compute the value of GM_{sun} from the known values of G and the solar mass.
2. Using planetary distance data, calculate orbital velocity using

$v = \sqrt{(GM / r)}$

3. Calculate orbital size using the circumference of the orbit

Orbit size = $2\pi r$

4. Compute orbital period using

$T = (2\pi r) / v$

5. Convert the period into years and compare the results with known planetary values.

The calculations are first performed for three representative planets:

- Mercury
- Earth
- Neptune

These planets were chosen because they span the inner and outer Solar System and clearly illustrate how orbital speed decreases with increasing distance from the Sun.

🧠 Discussion and Key Observations

After completing the calculations, students are asked to examine the planetary data and identify patterns.

Important observations include:

- Planets closer to the Sun move faster.
- Orbital speed decreases with distance from the Sun.
- Orbital period increases rapidly with distance.

Students are encouraged to examine the relationship between orbital period T and orbital radius r. When the data are analyzed, they reveal the relationship

$$T^2 \propto r^3$$

This is Kepler's Third Law.

The activity shows how Kepler's empirical law emerges naturally from Newton's theory of gravitation.

🧑‍🏫 Suggested Discussion Questions

Teachers may guide discussion with questions such as:

- Why must planets experience acceleration even if their speed is constant?
- What force provides the centripetal acceleration for planetary motion?
- Why does orbital speed decrease with distance from the Sun?
- Why do outer planets have such long orbital periods?
- How does Newton's law explain Kepler's Third Law?

Encourage students to think about how a single constant GM_{sun} can determine the motion of every planet.

🪄 Mathematical Insight

Combining the orbital velocity formula

$$\mathbf{v = \sqrt{(GM_{sun} / r)}}$$

with the period formula

$$\mathbf{T = 2\pi r / v}$$

leads directly to

$$\mathbf{T^2 = (4\pi^2 / GM_{sun})\, r^3}$$

This equation is the Newtonian derivation of Kepler's Third Law.
Students may not derive this formula explicitly during the activity, but they should recognize the relationship through the data.

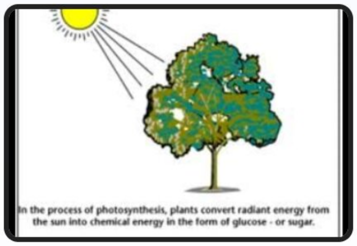
🧠 Reflection

One of the most powerful ideas students encounter in this lesson is that a single number — GM_{sun}— determines the motion of the entire Solar System. Once this constant is known, the orbital speeds and periods of all planets can be predicted from their distances from the Sun.
This illustrates the remarkable predictive power of Newton's laws and provides a bridge between mathematics, physics, and astronomy.

🏁 A Memorable Moment from the Math Circle

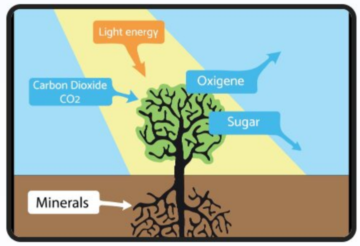
During the opening discussion about fire, students explored the idea that fire is the release of stored solar energy in chemical form. When asked the question, "What is fire?" Rooz immediately answered, "The Sun." Cyrus smiled, and the moment nicely captured the idea that the energy driving many processes on Earth ultimately originates from the Sun.
This observation provided a natural conceptual link between the opening discussion of combustion, and the later exploration of how the Sun governs the motion of the planets.

Adventure 15 Story – Fire, the Sun, and the Making of Civilization



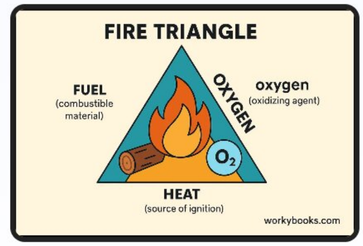
Plants convert sunlight into stored chemical energy.

Through photosynthesis, solar energy is stored in sugars and plant tissue.



Trees store solar energy in chemical bonds.

Carbon from the air becomes part of wood, locking in energy from the Sun.



🔥 Fire releases sunlight stored in trees.

Combustion converts stored chemical energy into heat, light, and carbon dioxide.

Of all the discoveries humans have made, few were as transformative as fire. The wheel helped us move things. Writing helped us remember things. But fire helped us survive long enough to invent both.

Early humans probably first encountered fire through lightning strikes or volcanic eruptions. At first it must have been terrifying — a roaring, glowing force that consumed forests and turned night into flickering day. But eventually curiosity overcame fear. Humans learned not just to use fire, but to make it. This single step changed the trajectory of our species.

Cooking food made nutrients easier to absorb. Anthropologists believe that cooked diets helped support the growth of larger human brains. Fire also extended the day, allowing people to gather, talk, plan, and imagine. Around campfires, language deepened, stories formed, and cultures began.

Fire transformed materials too. Clay hardened into pottery and bricks, making permanent settlements possible. Sand melted into glass. Metals softened in furnaces and could be shaped into tools and weapons. The rise of civilizations — from Mesopotamia to Persia to Rome — depended on the controlled power of flame.

Humans are, as far as we know, the only creatures to deliberately create fire. Myths filled the gap in understanding. Many cultures imagined gods or dragons breathing flame, symbols of power over nature. In a poetic sense, these myths were not entirely wrong.

Science later revealed that fire is a chemical process. When wood burns, atoms rearrange, releasing stored energy as heat and light. But this only pushes the mystery one step further:
why was energy stored in the wood at all?

The answer leads us beyond Earth, to the Sun. Plants capture sunlight through photosynthesis, storing solar energy in chemical bonds. Trees are, in effect, living batteries charged by starlight. When wood burns, the Sun's ancient energy is released again.

Fire is sunlight stored in trees.

This realization connects everyday experience to cosmic physics. The same Sun that warms your face and fuels a forest fire also governs the motion of planets. Its gravity shapes orbits, seasons, climates, and the architecture of the Solar System itself.

In this adventure, you will explore how a single quantity — the gravitational strength of the Sun — determines how worlds move through space. From the first campfires to the paths of distant planets, the story of science is a story of understanding energy, motion, and the laws that connect them.

🌠 Adventure 15 – Student Activity Sheets Solutions

🔹 Step 1 — Compute GM_{sun}

Quantity	**Value**
Gravitational constant G	$6.67 \times 10^{-11}\ N \cdot m^2/kg^2$
Mass of the Sun M_{sun}	$2 \times 10^{30}\ kg$
Result GM_{sun}	$1.33 \times 10^{20}\ m^3/s^2$

📊 Activity 1 — Orbital Velocity and Period

Planet	**Distance from Sun r (m)**	**Orbital Velocity v (m/s)**	**Orbit Size 2πr (m)**	**Period T (s)**	**Period (years)**
Mercury	5.79×10^{10}	4.80×10^4	3.64×10^{11}	7.58×10^6	0.24
Earth	1.496×10^{11}	2.99×10^4	9.40×10^{11}	3.15×10^7	1.00
Neptune	4.495×10^{12}	5.45×10^3	2.82×10^{13}	5.18×10^9	164

🔥 Activity 2 — Observations

Question	**Solution**
How does orbital speed change with distance?	Orbital speed decreases as distance increases.
What happens to orbit size?	Orbit size increases.
What happens to orbital period?	Period increases strongly.
Relationship between T and r	$T \propto r^{3/2}$

Adventure 15 – Student Activity Sheets

Activity 1. The Sun Controls the Solar System

In this activity, you are going to take control of the Solar System. Using Earth's motion, you will calculate a single powerful number — GM_{Sun}— that determines how strongly the Sun governs everything around it. Then you will use that number to compute the orbital speeds and periods of 3 planets. Then you will use the full planetary data and charts to make your observations.

We assume circular orbits for all planets then the orbital velocity: $\boldsymbol{v} = \sqrt{\frac{\boldsymbol{GM_{\text{Sun}}}}{\boldsymbol{r}}}$

So the orbital velocity can be calculated for all the planets once we calculate $\boldsymbol{GM_{\text{Sun}}}$.

G is Newton's gravitational constant: **G = 6.67 × 10^{-11} N m^2 / kg^2**

M_{sun} is the Mass of the Sun: **M_{sun}= 2 × 10^{30} kg**

Compute GM_{sun} = ________________________________

Table 1 — Orbital Velocity and Period for 3 Planets (No Decimals Unless it is a number < 1

Planet	Distance from Sun (m)	Orbital Velocity (m)	Orbital Size (m)	Period (second)	Peroid (Year)
Variable/Formula	r	$v = \sqrt{\frac{GM_{\text{Sun}}}{r}}$	$C = 2\pi r$	$T = 2\pi r/v$	$\frac{T}{31,557,600}$
Mercury	5.8×10^{10}				
Earth	1.5×10^{11}				
Neptune	4.5×10^{12}				

Activity 2. Use the Planetary Data Tables and Charts to answer the following questions:

1. As distance from the sun increases, what happens to orbital speed (explain)?

2. As distance from the sun increases, what happens to all the other variables (explain)?

3. What is the relationship between **Period T** and the variable $\mathbf{r}^{3/2}$?

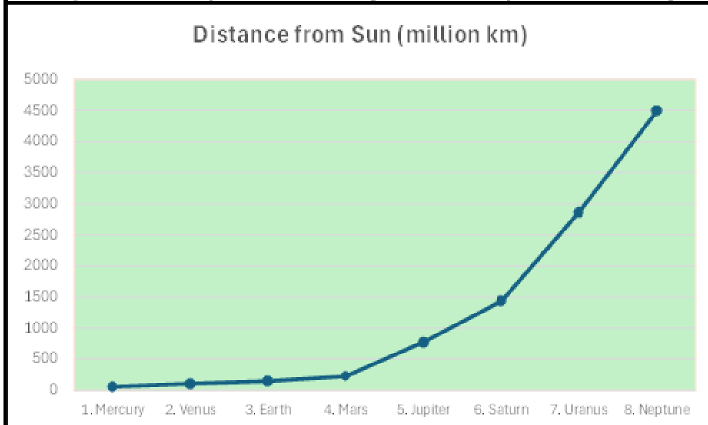
4. Using the formula for **v** and **T** can you explain the relation in (3)

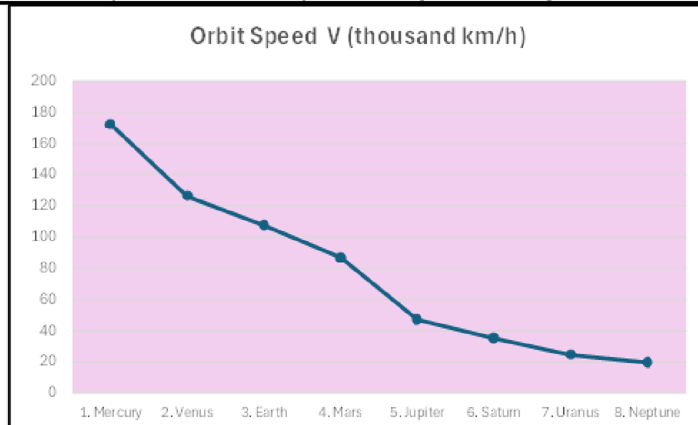
$$v = \sqrt{\frac{GM_{\text{Sun}}}{r}} \qquad \boldsymbol{T = 2\pi r/v}?$$

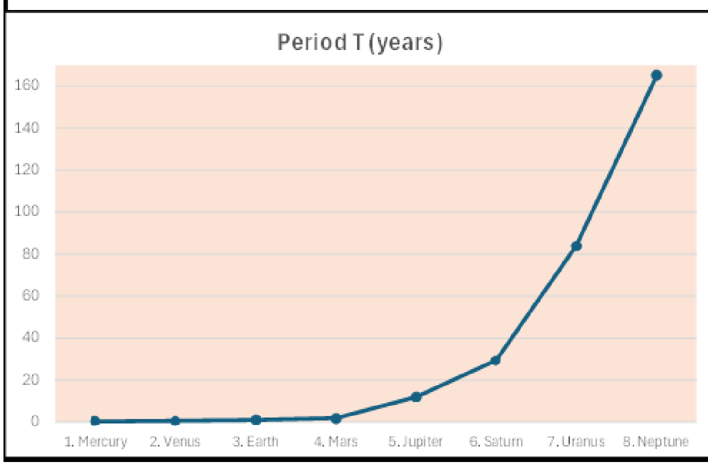
5. What other interesting relations or facts do you find in this data and charts?

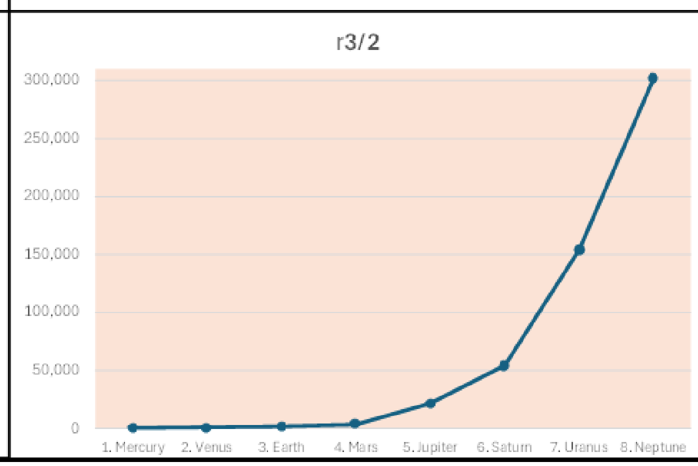
Planetary Data Comparisons

Planet	Distance from Sun (million km)	Distance from Sun (light-min)	Orbit Size (million km)	Orbit Speed (km/s)	Orbit Speed V (thousand km/h)	Period T (years)	$r^{3/2}$	$T/r^{3/2}$
1. Mercury	57.9	3.2	364	47.9	172	0.24	441	1,836
2. Venus	108.2	6	680	35	126	0.62	1,125	1,815
3. Earth	149.6	8.3	940	29.8	107	1	1,830	1,830
4. Mars	227.9	12.7	1,432	24.1	87	2	3,440	1,830
5. Jupiter	778.6	43.3	4,891	13.1	47	12	21,726	1,832
6. Saturn	1,433	79.6	9,005	9.7	35	29	54,246	1,841
7. Uranus	2,872	159.6	18,048	6.8	24	84	153,913	1,832
8. Neptune	4,495	249.7	28,254	5.4	19	165	301,366	1,829









The original explorers of these Adventures:

Rooz (9), Cyrus (11), and Marc (11)

Every lesson, activity, chart, and discussion in this book was tested during weekly Wednesday Math Circles. Their questions, discoveries, and insights helped shape the Adventures and demonstrated that young students can understand the fundamental ideas of calculus long before they encounter its formal techniques.

About the Author

Behrouz B. Aghevli, PhD (Dr. Super) earned degrees in Physics, Mathematics, and Statistics from Occidental College and Northwestern University. Over a career spanning university teaching, educational publishing, software development and Business Intelligence, he also authored mathematics materials, developed educational software, and presented at (NCTM) conferences.

Beginning in the early 1990s, while exploring mathematics with his daughter, Dr. Super developed a lifelong passion for creating hands-on, visual approaches to mathematics. Over the next three decades, he worked closely with teachers and students, developing classroom activities, educational software, and mathematics programs spanning kindergarten through high school.

A pioneer in Virtual Manipulatives and hands-on mathematics education, he holds four U.S. patents, including Virtual Manipulatives, Factor Blocks, Terrific Triangles, and the Fractal Mathematics Kit.

In 2022, inspired by his grandchildren Cyrus and Rooz, he returned to teaching through weekly Math Circles. These explorations ultimately led to the creation of the Adventures series and the DiVA (Distance–Velocity–Acceleration) charts and approach to learning calculus through visualization, motion, and discovery.