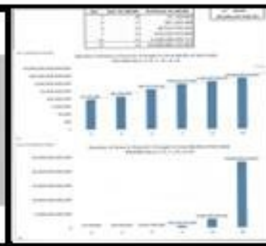
Math Circle 2024-2025 Cyrus, Rooz and Marc

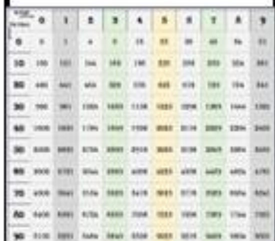

1. Algebra Blocks 1

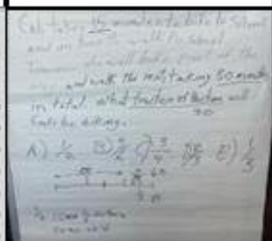

2. Algebra Blocks 2

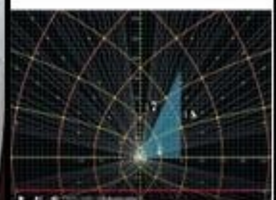
3. Algebra Blocks 3

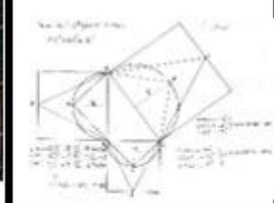

4. Logarithms

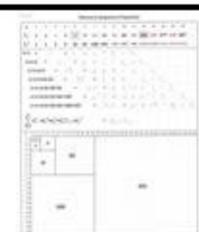
5. Math League 1


6. Math League 2


7. Math League 3


8. Pythagorean Triples


9.Pythagorean Triples 2

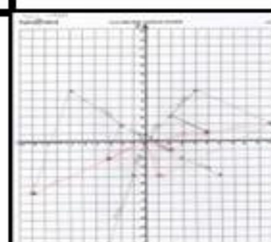

10. Fibonacci Numbers Proof by Induction

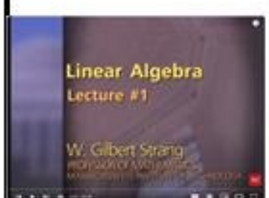

11.Math League Team Round Strategy 1


12.Math League Team Round Strategy 2


13. Math League Percentages 1


14. Math League Percentages 2


15. Essence of Linear Algebra 1

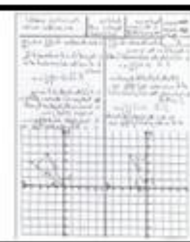


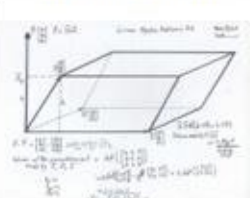
16. Linear Combinations Basis Vectors , Span


17. Linear Transformations & Matrices


18. Matix Multiplication Determinants


19. Inverse Matrices Column and Row Spaces


20. Dot Products and Duality


21.Linear Algebra Cross Product


22.Graduation and Celebration


22.Graduation and Celebration


22.Graduation and Celebration


22.Graduation and Celebration

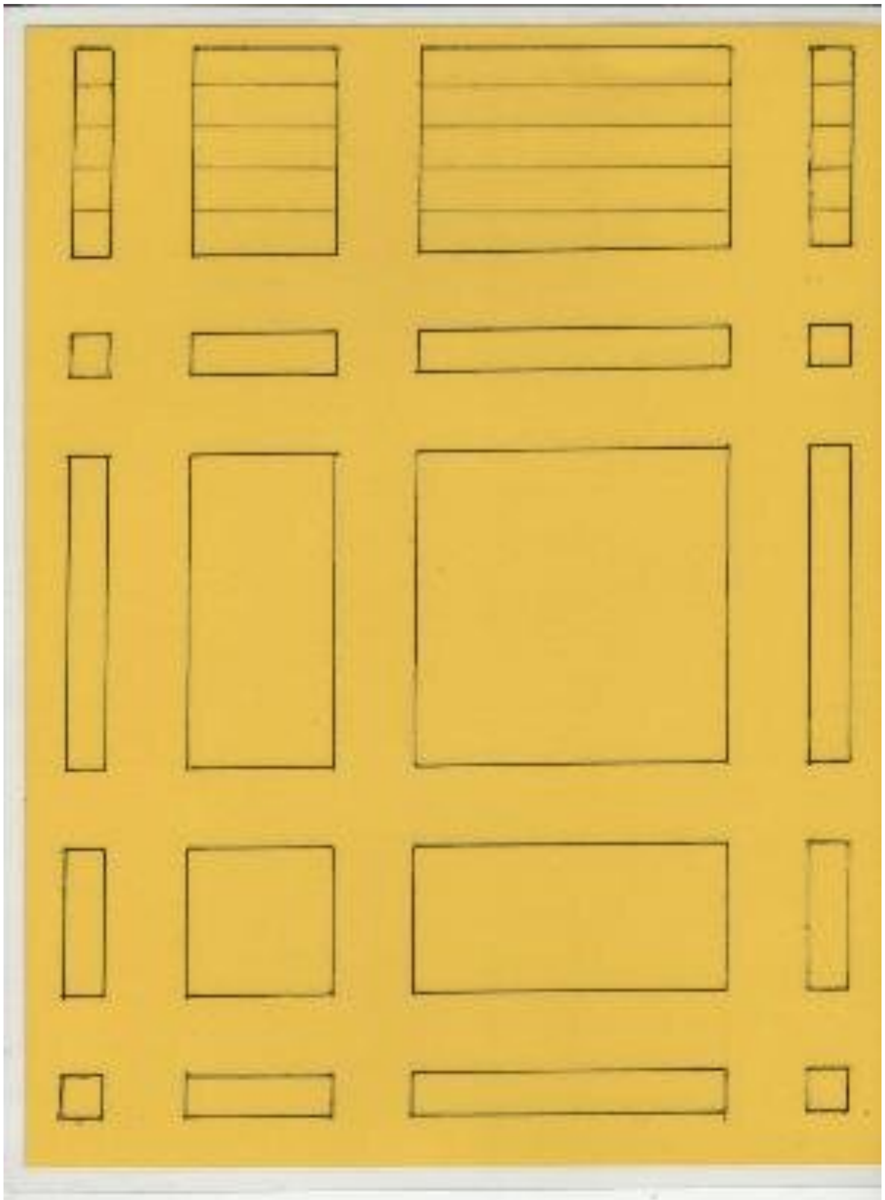
Table of Contents

Math Circle 1 Algebra 1 8/13/2024 1:50-3:00

Cyrus, Marc and Dar (Rooz crashed the first 10 minutes)

This year we are going to start with an Algebra book from someone I know from NCTM conferences. I always wanted to use his book as it introduced Algebra Lab Gear that is a great method to make Algebra more concrete by using a manipulative. The book is designed for Algebra 1 and it was meant to be covered in one academic year generally in 9^{th} or 10^{th} grade but the consensus is that it could not be covered in 1 year. I intend to use the final section of some of the chapters that are titled: "Essential Ideas". I am attaching the one for the first chapter. This is also the link for the book in PDF "Algebra: Themes, Tools and Concepts" that you may wish to download.

We first played a simple game that I have invented using Lab Gear. We played it as a community game. Rooz also crashed the party and played as he knows it. You roll two dice that their product will give you the values 1 or 5 or an expression with x and y (x, y, x2,y2,xy2, x2y, x3, y3) then you have to put that piece on the following board. It was good for Marc as he had not seen Lab Gear before (I think).



Then Cyrus and Marc covered the first 12 problems of the Essential Ideas for the first chapter together. It went well and they discussed and setup the problems together. Finally they got a little stuck of problems 14 and 15 that go together. We may finish the rest of the problems next time or just discuss them. Their worksheets are below. As always Cyrus was encouraged to work faster and Marc to use better penmanship. They have improved from last year and we discussed the fact that they needed to show all the steps otherwise they may not get full credit on exams. Marc must have heard this in class because he supported it strongly.

It was a long session and a very good start for the Math Circle for 2024-2025.

amazon basics

1. 15
2. 75
3. 45
4. $x^3+x^2+3x+5=8+4+6+5=23$
5. $x^2y+x^3+xy+x+5=v$
   a. $x=0 \quad y=2 \quad v=0+0+0+0+5=5$
   b. $x=5 \quad y=1 \quad v=25+125+5+5+5=165$
   c. $x=2 \quad y=3 \quad v=12+8+6+2+5=33$

6. $xy^2+y^2+2y+x^2+3=z$

   a. $z=0+4+4+0+3=11$
   b. $z=5+1+2+25+3=35$
   c. $z=18+9+6+4+3=40$

7. $2y^2+2x+3$
8. x^2+3x+3
9. $3x^2+2y+5$
10. $3x^2+2x+5$
11. $3x^2+2xy+y$
12. x^3+3x^2+8x+9
13. a. $x^2 \cdot x=x^3$
   b. $3x \cdot x=3x^2$
   c. $x^2 \cdot x^2 \cdot x^2 \cdot x^2=x^8$
   d. $y^6=y^6$
   e. $2xy+3xy=5xy$

amazon basics

1. 15
2. 75
3. 45
4. x^3+x^2+3x+5
$=2^3+2^2+3(2)+5$
$=8+4+6+5$
$=12+11$
$=23$
5. $5+x+xy+x^3+x^2y=V$

a. $x=0 \quad y=2 \quad V=5$
$5+0+0 \quad 5+0+0+0+0$
$=V \quad =v$
$=5$

b. $x=5 \quad y=1 \quad z=65$
$5+5+5+125+25$
$=z$
$=165$

c. $x=2 \quad y=3 \quad W=33$
$5+2+6+8+12$
$=W$
$=33$

6. $3+x^2+2y+y^2+x(y)^2=z$
a. $z=3+0+4+4+0$
$=11$
b. $z=3+25+2+1+5$
$=$

c. $z=3+4+6+9+18$
$=13+27$
$=40$

7. $3+2x+2y^2$

8. $3+3x+x^2$

9. $2y+3x^2+5$

10. $10+3x^2+2x$

11. $3x^2+y+2xy$

12. x^3+3x^2+8x+9

13.

a. $x^2 \cdot x$
b. $3x \cdot x=3x^2$
c. $x^2 \cdot x^2 \cdot x^2 \cdot x^2=x^2$
d. $y^6=y^6$
e. $2xy+3xy=5xy$

14.

Math Circle 2 Algebra 2 8/21/2024 1:50-3:00

Cyrus, Marc and Dar

We continued with the problems in the first chapter that are titled: "Essential Ideas" from the PDF "Algebra: Themes, Tools and Concepts".

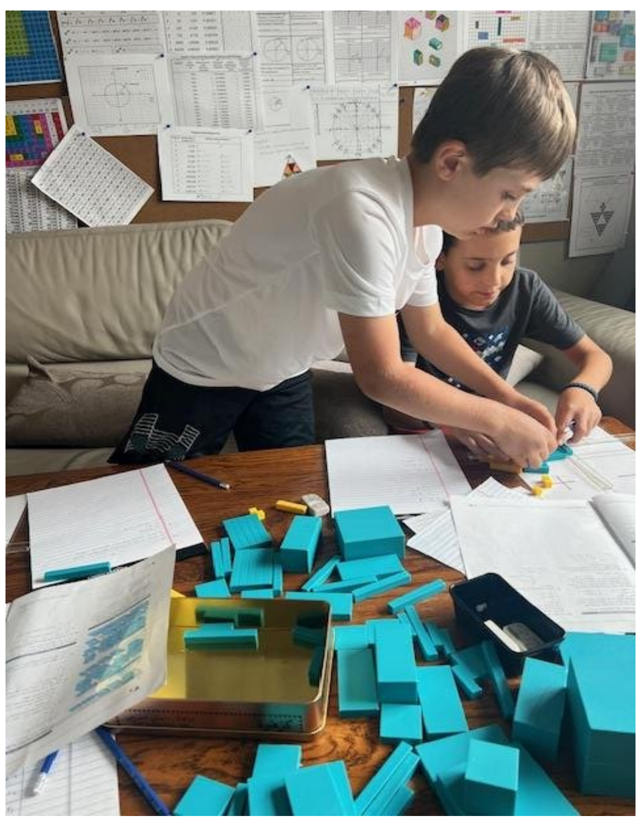
I had prepared a table for problems 13, 14, 15, 16 and 17 to keep them moving. Cyrus and Marc had decided to use their own numbering system so it would not correspond exactly to the handout from the book.

With the tables that I made for them they had an easy time solving problems 13 to 16. These exercises helped them with adding and multiplying variables and coefficients. They worked well together. I have attached their work below.

Next, we did problems 17 and 18 that dealt with order of operations and they both used the lab gear and could see how parenthesis should be used and were used.

Next, they worked on problems 19 to 21 that dealt with Area and multiplication of variables and constants. They used the lab gear a lot and had a lot of fun with these problems. In particular for problems 20 and 21 where they had to make the rectangles and then find the missing side(s). We discussed the four quadrants that Marc was familiar with and Cyrus had seen for the first time. Cyrus then figured out with help from Marc that if he made both multiplicands negative the result would be the same as with them positive

It was a great session that was both hands on and also, they did a lot of introductory algebraic operations on paper. As you can see my(our) continuous encouragement of Marc to write more clearly is paying off and he is more and more careful. However, Cyrus goes at his own speed, and I must see how I can get him to go a bit faster. They work well together, and I enjoy watching them work together. I did a short video that I will attach on maybe just text to you.



13, 14

x, y	$2 + x + y$	$2 + xy$	$2x + y$	$2xy$
1, 1	4	3	3	2
1, 2	5	4	4	4
0, 1	3	2	1	0
2, 2	6	6	6	8
3, 3	8	11	9	18

16

x, y	$x + y^2$	$x^2 + y$	$x^2 + y^2$	$(x+y)^2 = x^2 + y^2 + 2xy$
1, 1	2	2	2	4
1, 2	5	3	5	9
2, 3	11	7	13	25

Cyrus

13, 14

x, y	$2 + x+y$	$2+xy$	$2x + y$	$2xy$
1,1	4	3	3	3
1,2	5	4	4	4
0,1	3	2	1	0
2 2	6	6	6	8
3,3	8	11	9	18

16

x, y	$x+y^2$	x^2+y	x^2+y^2	$(x+y)^2$ $=x^2+2xy+y^2$
1,1	2	2	2	4
1,2	5	3	5	9
2,3	11	7	13	25

Marc

14. a. 1

b. $y=0$ $2+5y > (2+5)y$ because $2>0$

c. $y=2$ $2+5y < (2+5)y$ because $14>12$

15. a. $3 \times 5 = 15$

b. $3x$

c. $3 \cdot (x+5) = 3x+15$

d. $x \cdot (x+5) = x^2+5x$

16. a. $4 \cdot x = (-4) \cdot (-x) = 4x$

b. $(4)+(2x) \cdot 2x = 2+x \cdot 4x = (-)(4+2x) = (-4)\,2x = 4x^2+8x$

c. $x+y \cdot 4x+3 = 6x^2+3xy+9x$

17. a. $x \cdot (x+y) = x^2+xy$

b. $(y+1)5 = 5y+5$

c. $(2x+3)y = 2xy+3y$

d. $2x \cdot (3x+y+2) = 6x^2+4x+2xy$

Cyrus

14 a. 1

b $2 > 0$

$y=0$ $2+5y$ $7y$
$=2+0$ $=0$
$=2$

c. $14 > 12$

$y=2$ $2+10$ $7y$
$=12$ $=14$

15. a $3(5)=15$

b, $3(x)=3x$

c. $3(x+5)$
$=3x+15$

$x \cdot (x+5)$
d. $=x^2+5x$

a $4 \cdot x = 4x$ $(-4x) \cdot (-1) = 4x$

b $4x^2+8x$
$4+(2x) \cdot 2x = 2+x \cdot 4x = (4+2x)(-)2x = 4x^2+8x$

C. $x+y \cdot 4x+3 = 6x^2+3xy+9x$

amazon basics

16. $x(x+y)=x^2+xy$

$5(y+1)=5y+5$

$y(2x+3)=2xy+3y$

$2x(y+3x+2)$

Marc

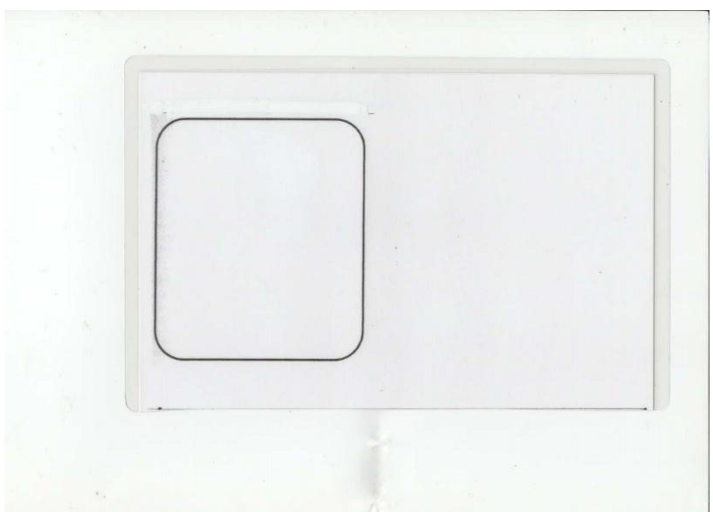
Math Circle 3 Algebra 3 8/29/2024 1:50-3:00
Cyrus, Marc and Dar

We finished the problems in the first chapter that are titled: “Essential Ideas” from the PDF “Algebra: Themes, Tools and Concepts” and started the problems on the second chapter “Essential Ideas” mostly dealing with the concept of the negative sign and its 3 meanings.

Problems 19-22 in the first chapter deal with Area and multiplication of constants and variables and expressions. Cyrus and Marc used the Algebra Lab Gear (ALG) first and then wrote down the answers. For some of the problems they wanted to just skip the ALG and just write the answers, but I encouraged them to use the ALG either before or if not after they had found the answers to check their results. This gives them a much better understanding of how the multiplication of variables and expressions works. This is a more time-consuming effort but well worthed. Even in Marc work on paper you can see that he has slowed down. Cyrus goes at his own speed, and I have not been able to encourage him to write a little faster and maybe sloppier.

Next, we started the discussion for the second chapter about negative sign and its 3 meanings: negative number, opposite value, Subtraction. They did problems 1, 2, 3 and used the special pages where the negative numbers and variables go into a separate part.

It was a good session, and they worked together well and discussed what had to be done with me, when they were using the ALG. They keep asking to do Logarithms again so I may do a review of what they did last time in their next session.



Negative variables and numbers go into the rounded square. Equivalent values and variables from inside and out side of the square cancel each other.

Cyrus

14\. a. 1

b. $y = 0$ $\quad 2 + 5y > (2 + 5)y$ because $2 > 0$

c. $y = 2$ $\quad 2 + 5y < (2 + 5)y$ because $14 > 12$

15\. a. $3 \times 5 = 15$

b. $3x$

c. $3 \cdot (x + 5) = 3x + 15$

d. $x \cdot (x + 5) = x^2 + 5x$

16\. a. $4 \cdot x = (-4) \cdot (-x) = 4x$

b. $(4) + (2x) \cdot 2x = 2 + x \cdot 4x = (-)(4 + 2x) \cdot (4)\, 2x = 4x^2 + 8x$

c. $x + y \cdot 4x + 3 = 6x^2 + 3xy + 9x$

17\. a. $x \cdot (x + y) = x^2 + xy$

b. $(y + 1)5 = 5y + 5$

c. $(2x + 3)y = 2xy + 3y$

d. $2x \cdot (3x + y + 2) = 6x^2 + 4x + 2xy$

18\. $P = 8x$

$W = 2$ $\quad L = 2$ $\quad A = 4x^2$

$W = 1$ $\quad L = 3$ $\quad A = 3x^2$

$P = 6x + 2y$

$W = 3x$ $\quad L = y$ $\quad A = 3xy$

$W = 2x$ $\quad L = x + y$ $\quad A = 2x^2 + 2xy$

$P = 4x + 4y$

$W = 2x$ $\quad L = 2y$ $\quad A = 4xy$

$W = x + y$ $\quad L = x + y$ $\quad A = x^2 + y^2 + 2xy$

19\. $a = 7$ $\quad b = 9$ $\quad A = 31.5$

$a = 4x$ $\quad b = y$ $\quad A = 2xy$

amazon basics

1. a. -2 N

   b. $x-2$ S

   c. $-(2+2x)$ O

   d. -2 N

2. a. $-x$

   b. -2

   c. 2

   d. x

   e. $-x-2$

   f. $-x+2$

3. a. $(y^2+\cancel{x^2}-3y)+(y+3x^2-\cancel{x^2}) = 3x^2+y^2+2y$

   b. $\cancel{x}+25-\cancel{xy}-y^2+\cancel{xy}-y-\cancel{x} = -y^2-y+25$

4. a. $4-x^2-5x-3x-2$

Marc

$a=7 \quad b=9$

16. $x(x+y)=x^2+xy$

18. $a=31.5$

$5(y+1)=5y+5$

b. $2xy$

$y(2x+3)=2xy+3y$

$a=4x \quad b=y$

$2x(y+3x+2)$

17.

a. $L=2x$
$W=2x$
$A=4x^2$

$L=x$
$W=3x$
$A=3x^2$

b
$L=3x$
$W=y$
$A \; 3xy$

$L=2x$
$W=x+y$
$A=2xy+2x^2$

c. $L=2y$
$W=2x$
$A=4xy$

$L=x+y$
$W=x+y$
$A=x^2+y^2+2xy$

amazon basics

1. a. N b. 0
c. 5 d. 0

2. a. x b. -2
c. 2 d. x
e. -x -2 f. -x+2

3. a. $(y^2 \cancel{+x^2} \cancel{-3y}) + (\cancel{y} + 3x^2 \cancel{-x^2})$
$-2y$
$y^2 + 3x^2 + 2y$

b. $\cancel{x} + (25 \cancel{-yx} - y^2) + (\cancel{xy} - y \cancel{-x})$
$-y^2 - y + 25$

c. $(\cancel{4} - x^2 \cancel{-5x}) \cancel{-3x} \cancel{-2}$
$-x^2 - 8x + 2$

Math Circle 4 Logarithms 9-11-2024 1:50-3:30

Cyrus, Marc and Dar

As the boys have been asking about a repeat of a session on the Logarithms I decided to repeat what we had done with the exercises from the [Logarithm Fundamentals Video from 3 Blue one Brown Live](). I watched the live myself and did the exercises and realized how much I had to review. I chose parts of the live, approximately 30, minutes and the rest of the time we did the exercises As the live goes too slow for them to watch all of it I did some of the explanations myself and made a single sheet with all the questions for them.

They wanted all the rules to begin with, but I refused to give it to them and wanted them to discover the rules by doing the exercises that they did. I basically wanted to do what Grant did in the video and gave the same questions to them I also added some myself. Starting with the basic fact that in base 10, Logarithms count zeros when you evaluate the powers of 10. I had already mentioned that I called Logarithms the “Power Spitter Functions” or simply “Spitters” and they liked my name better. Also spitting can be done for all bases for the powers of the base so it is more general.

I focused the session on the use of the Logarithmic scale and started with Grants explanation on how the COVID cases could be predicted using a Logarithmic scale. I had them watch this part of the Video and we discussed it. In particular the fact that if the growth of some thing is multiplicative you will end up with an exponential function and using the Logarithmic scale would allow you to predict values as opposed to a linear scale.

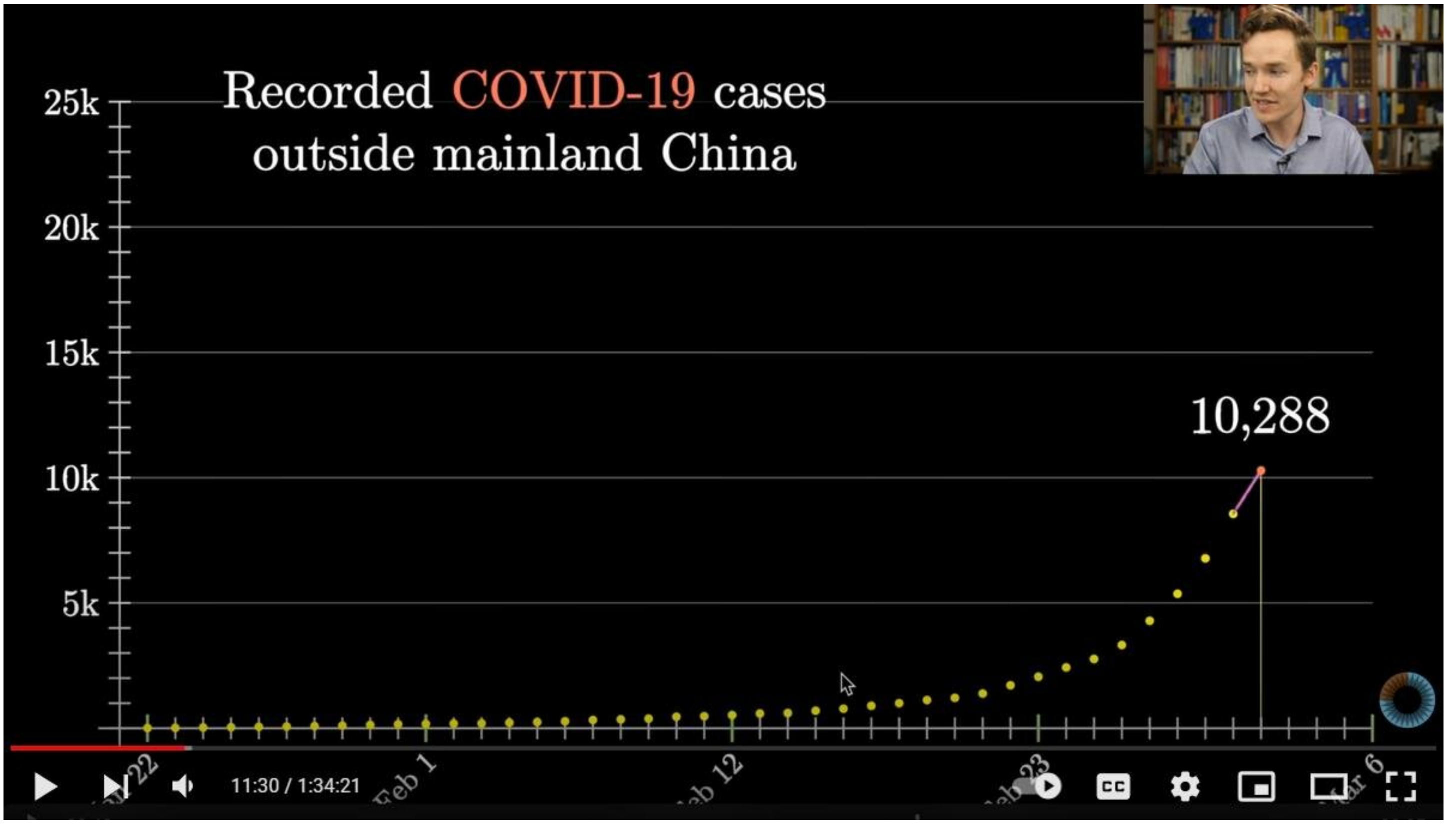
I also had them look and discuss the 99.9 divisibility chart by primes in the Pascal’s triangle that they had seen last time and try to make those estimates for other primes, say 17 is impossible with a linear scale but strait forward with the Logarithmic scale.

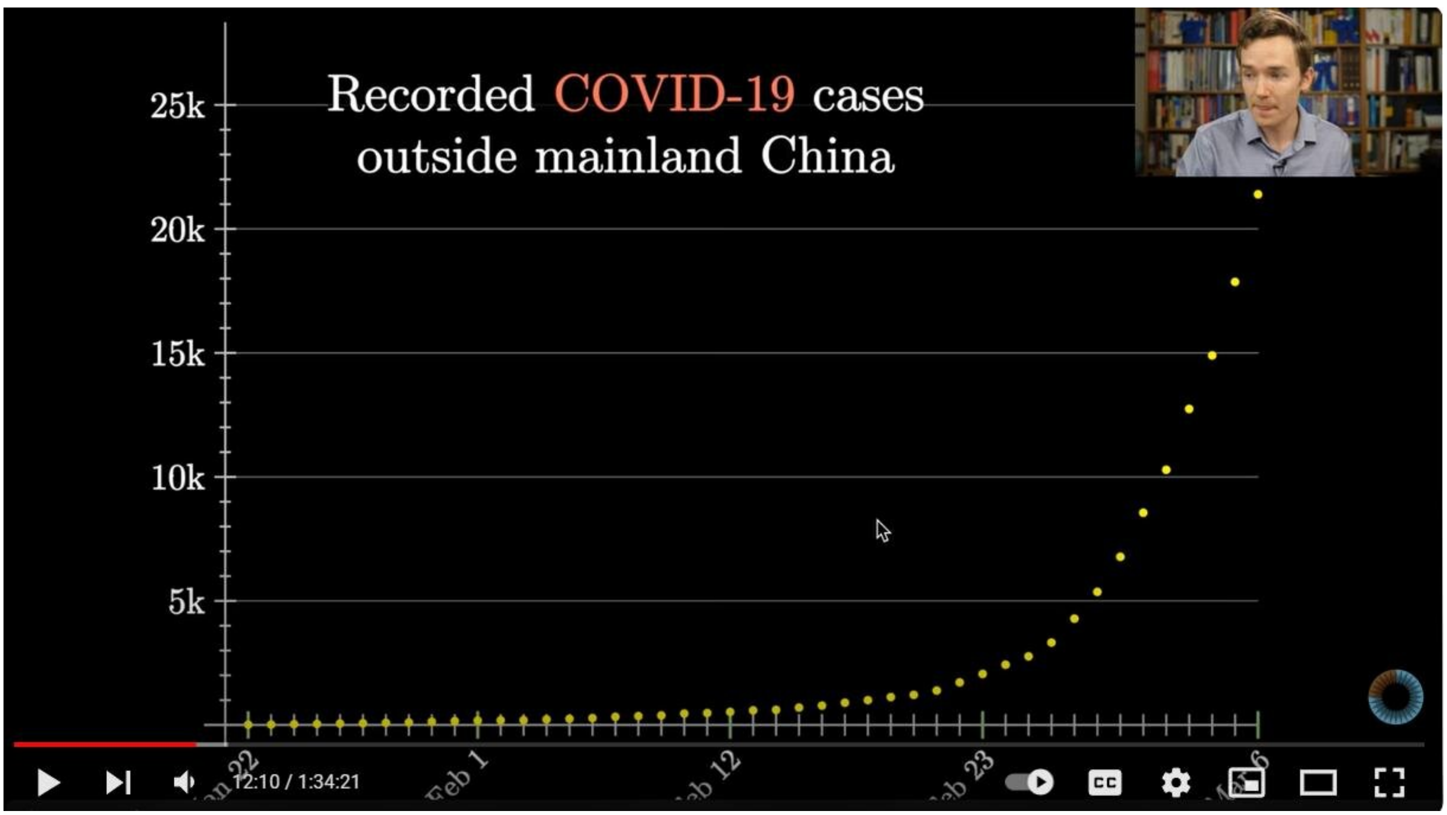
We also looked at the Richter Scale chart again and how at each step the equivalent force measured in TNT equivalent grows by a factor of 32 so it grows by a factor of 1K each 2 steps. We did not get to look at how this was exactly related to the change of base in logarithms, but we may discuss it next time.

Next. they did the problems on the sheet and figured out what rule applied to each one. I will share the rules with them next time and review them. I think they have a much better understanding of these rules and why Logarithms are important now.

We also watched and discussed the triangle method of relating power, root and logarithm as inverse functions.

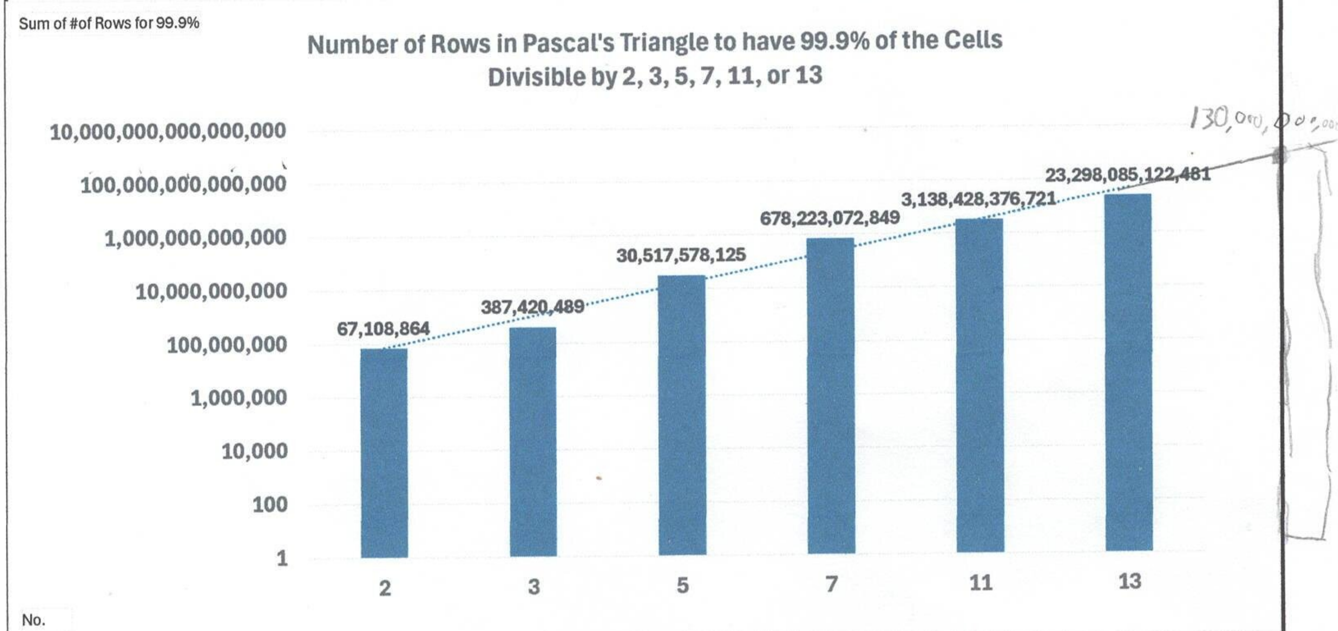
It was a very good session, and they got to play some foosball at the end but forgot to cover the table afterwards.

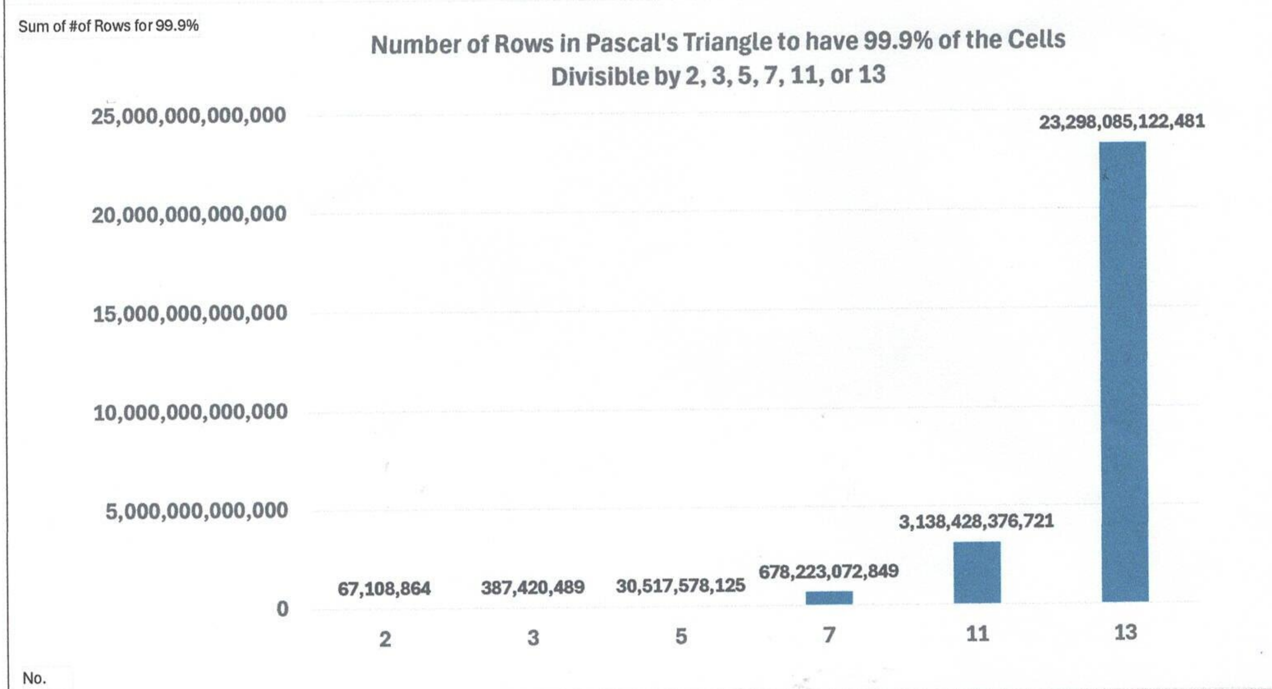

Recorded COVID-19 cases
outside mainland China
25k
20k
15k
10k
5k
10,288
Feb 1
11:30 / 1:34:21


Recorded COVID-19 cases
outside mainland China
25k
20k
15k
10k
5k
Feb 1
12:10 / 1:34:21

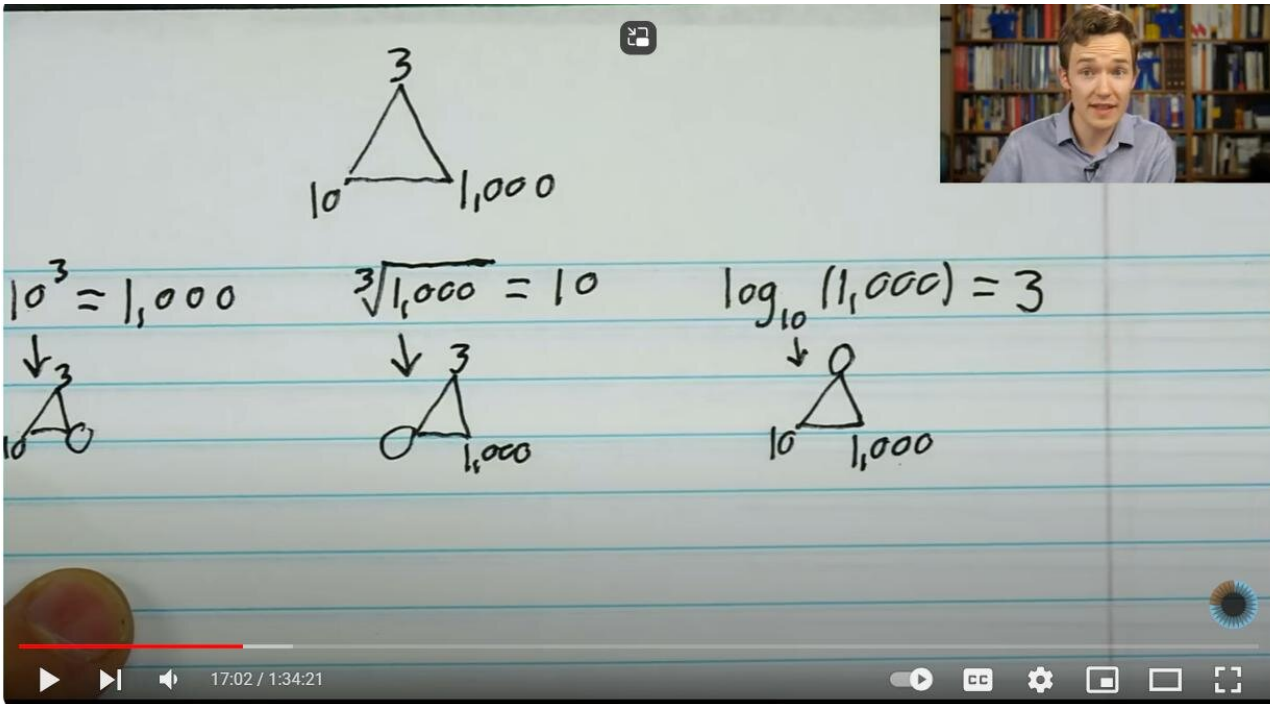
No.	Gen for 99.9%	#of Rows for 99.9%
2	25	67,108,864
3	17	387,420,489
5	14	30,517,578,125
7	13	678,223,072,849
11	11	3,138,428,376,721
13	11	23,298,085,122,481

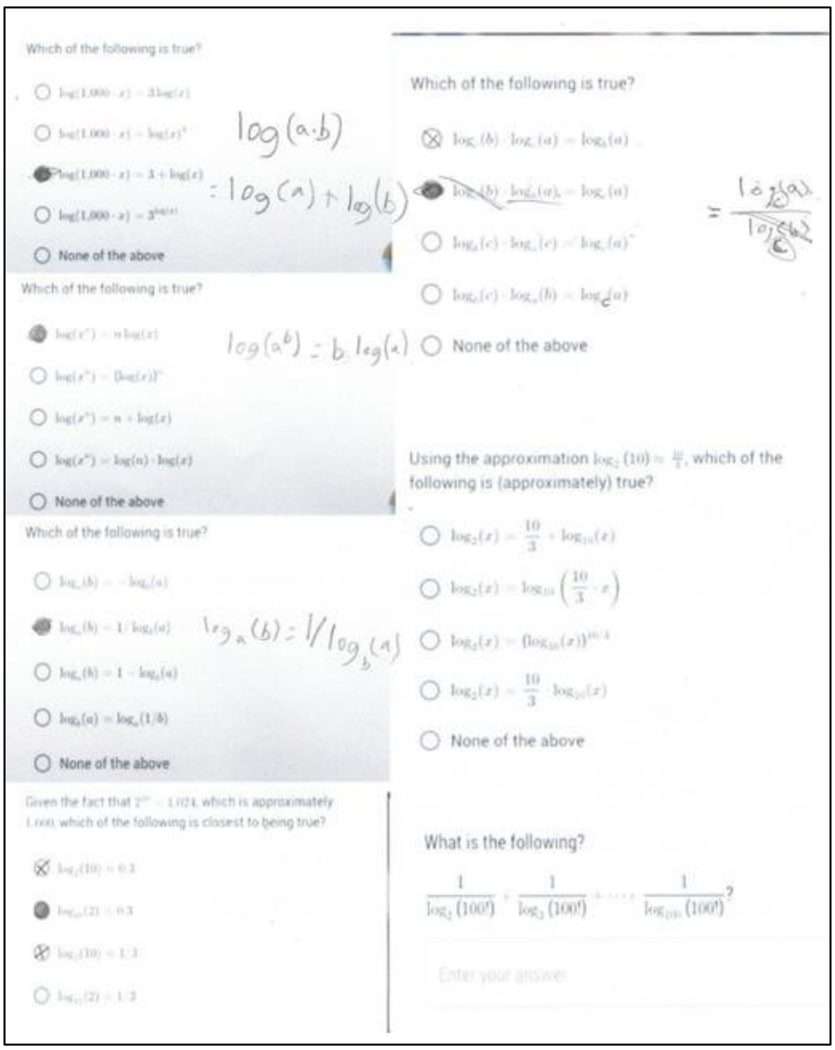
17	99.9%
582,622,237,229,761	





Richter Approximate Magnitude	Approximate TNT for Seismic Energy Yield	Joule equivalent	Example
0.0	1 kg (2.2 lb)	4.2 MJ	
0.5	5.6 kg (12.4 lb)	23.5 MJ	Large hand grenade
1.0	32 kg (70 lb)	134.4 MJ	Construction site blast
1.5	178 kg (392 lb)	747.6 MJ	WWII conventional bombs
2.0	1 metric ton	4.2 GJ	Late WWII conventional bombs
2.5	5.6 metric tons	23.5 GJ	WWII blockbuster bomb
3.0	32 metric tons	134.4 GJ	Massive Ordnance Air Blast bomb
3.5	178 metric tons	747.6 GJ	Chernobyl nuclear disaster, 1986
4.0	1 kiloton	4.2 TJ	Small atomic bomb
4.5	5.6 kilotons	23.5 TJ	
5.0	32 kilotons	134.4 TJ	Nagasaki atomic bomb (actual seismic yield was negligible since it detonated in the atmosphere. The Hiroshima atomic bomb was 15 kilotons) Lincolnshire earthquake (UK), 2008
5.4	150 kilotons	625 TJ	2008 Chino Hills CA earthquake
5.5	178 kilotons	747.6 TJ	Little Skull Mtn. earthquake (NV, USA), 1992 Alum Rock earthquake (CA, USA), 2007
6.0	1 megaton	4.2 PJ	Double Spring Flat earthquake (NV, USA), 1994
6.5	5.6 megatons	23.5 PJ	2010 Ferndale CA earthquake
6.7	16.2 megatons	67.9 PJ	Northridge earthquake (CA, USA), 1994
6.9	26.8 megatons	112.2 PJ	San Francisco Bay Area earthquake (CA, USA), 1989
7.0	32 megatons	134.4 PJ	2010 Haiti earthquake
7.1	50 megatons	210 PJ	Energy released was equivalent to that of Tsar Bomba, the largest thermonuclear weapon ever tested.
7.5	178 megatons	747.6 PJ	Kashmir earthquake (Pakistan), 2005 Antofagasta earthquake (Chile), 2007
7.8	600 megatons	2.4 EJ	Tangshan earthquake (China), 1976
8.0	1 gigaton	4.2 EJ	Toba eruption 75,000 years ago; which, according to the Toba catastrophe theory, affected modern human evolution Shaanxi earthquake (China), 1556, San Francisco earthquake (CA, USA), 1906
8.5	5.6 gigatons	23.5 EJ	
9.0	32 gigatons	134.4 EJ	
9.2	90.7 gigatons	379.7 EJ	Anchorage earthquake (AK, USA), 1964
9.3	114 gigatons	477 EJ	Indian Ocean earthquake, 2004 (40 ZJ in this case)
9.5	178 gigatons	747.6 EJ	Valdivia earthquake (Chile), 1960 (251 ZJ in this case)
10.0	1 teraton	4.2 ZJ	Estimate for a 2 km (~1.2 mi) rocky meteorite impacting at 25 km/s (~55,000 mph)


3
10
1,000
$10^3 = 1{,}000$
$\sqrt[3]{1{,}000} = 10$
$\log_{10}(1{,}000) = 3$
17:02 / 1:34:21


Which of the following is true?
$\log(a \cdot b) = \log(a) + \log(b)$
None of the above
Which of the following is true?
$\log(a^b) = b \cdot \log(a)$
None of the above
Which of the following is true?
$\log_a(b) = 1/\log_b(a)$
None of the above
Given the fact that $2^{10} = 1{,}024$, which is approximately 1,000, which of the following is closest to being true?
Which of the following is true?
None of the above
Using the approximation $\log_2(10) \approx \frac{10}{3}$, which of the following is (approximately) true?
$\log_2(x) = \frac{10}{3} + \log_{10}(x)$
$\log_2(x) = \log_{10}\left(\frac{10}{3} \cdot x\right)$
$\log_2(x) = \frac{10}{3} \cdot \log_{10}(x)$
None of the above
What is the following?
$\frac{1}{\log_2(100!)} + \frac{1}{\log_3(100!)} + \cdots + \frac{1}{\log_{100}(100!)}$?
Enter your answer

Logarithm Rules

$$\log(ab) = \log(a) + \log(b)$$

$$\log(a^n) = n\log(a)$$

$$\log_b(a) = 1/\log_a(b)$$

$$\log_b(a) = \log_c(a)/\log_c(b)$$

Math Circle 5 Math League Intro 9-18-2024 1:50-2:00 and 3:00-3:50

Cyrus, Marc, Rooz, Adam, and Dar with Laura

Laura had asked me to do an intro to math league as all 4 kids are going to try it out. She let me borrow a test book she had. The material is on MathLeague.ORG and can be purchased. I did download the most recent test also.

I worked on some of the problems and prepared a set for both of them to do. Before they started Laura told them what they would do on the preliminary sessions and what was expected of them in terms of behaviour etc.

For Cyrus and Marc I gave them the first 40 questions and 15 minutes they finished in less than 10 and they did almost all correctly. For Rooz and Adam I gave them 19 questions and 10 minutes (Rooz continued a bit more) they also finished before the time was up and also got most of the problems correctly. This part of the test called number Sense has 80 problems each week that they have to do in them in 10 minutes (this is my undrestanding). This is a challenge even for me! They need to learn and relearn and use a lot of Math tricks for multiplying and dividing finding squares and cubes and square roots and cube roots and finding remainders for dividing large numbers by one digit numbers, changing numbers to different basis, using the Pythagorian theorem and basic algebra identeties, sequences and their sums, etc. They need to be good at rounding and estimating also. I made some notes on divisibility that I shared with them. There was a problem with Roman Numerals that they had trouble with as Cyrus and Rooz had not worked with it and Marc and Adam had seen them but still were not comfortable with them. I went over it with them.

Later I also took another part of the test called the Sprint Round with 30 multiple choice questions and this required a lot more thinking and I had to work on at least 10 of them pretty hard and got 4 wrong at least. This will be challenging for all of them in particular for the younger kids.

I think overall this will be a great challenge for all of them irrespective of if they get into the team(s) and I will try to go over all the methods that can help them solve these types of problems. This will be a significant help for them when they get to taking similar tests including SAT later.

Divisibility check
& Remainder (R)

2) If the number is odd R=1

3) Take 1 or 2 units off to make the sum divisible by 3 R=2
For example: 377 take 2 away 375

4) take 1 or 2 or 3 units to make the last two digits divisible by 4
(Remember the rule for divisibility by 4)
(odd+2 or 6, even+0,4,8)
For example for 56,939 take 3 away to make it 56,936 R=3

5) take 1,2,3, or 4 units away to make the last digit 0 or 5
For example for 694,324 take 4 units away to make it 694,320 R=4

6) take 1 to 5 units away to make the result even with sum of the number divisible by 3 R=5
For example for 5261 take 5 away to make it 5256

7) take 1 to 6 units off to make the number divisible by seven
Use (a-2b rule)
For example for 14,625 ignore 14
for 625 take 2 away to make it 623
62-6=56 ✓ R=2

8) Do the long division only using the remainder to find R. Use the last 3 digits

72,939 939
13
59
3 R=3

9) take 1 to 8 units to make the sum of the numbers divisible by 9
For example for 2357
Take 8 units away R=8
Alternatively add 1 unit and R=(9-1)=8

MATH LEAGUE.ORG

Number Sense 11820

Name ____

Grade ____

School ____

correct and incorrect must sum to last problem attempted

Place ID Sticker Inside This Box

Correct ____ Correct ____

Incorrect ____ Incorrect ____

Scorer Initials ____ Scorer Initials ____

1. $17 + 2017 =$ 2034
2. The hundreds digit of 31415 is 4
3. $14 + 41 =$ 55
4. $6 \times 34 =$ 204
5. The product of 9 and 6 is 54
6. $192 \div 6 =$ 32
7. Round 376 to the nearest ten. 380
8. The remainder of $52 \div 7$ is 3
9. $524 - 294 =$ 230
10. (estimate) $85 + 95 + 105 + 115 =$ 420
11. $11 \times 26 =$ 286
12. $9^2 =$ 81
13. $217 + 192 =$ 409
14. $50 \times 16 =$ 800
15. LXXXVIII in Arabic numerals is 88
16. $10 \times 71 =$ 710
17. The remainder of $245 \div 4$ is 1
18. $24 \times 15 =$ 360
19. $17 \times 23 =$ 391
20. (estimate) $1111 + 222 + 33 + 4 =$ 1230
21. $\frac{3}{5} =$ 0.60 (decimal).
22. If 1 mile is equal to 5280 feet then 2 miles are equal to 10,560 feet.
23. $25 \times 21 =$ 525
24. $5 \times 44 \div 20 =$ 11
25. The GCD of 16 and 12 is 4
26. $\frac{1}{4} + \frac{1}{3} =$ 7/12 (fraction).
27. $40 \times 22 =$ 880
28. $\frac{6}{9}$ in lowest terms is 2/3 (fraction).
29. $23^2 =$ 529
30. (estimate) $19 \times 20 \times 21 =$ 8,000
31. The perimeter of a square with a side of length 5 is 20
32. $22 \times 28 =$ 616
33. Eight minutes is 480 seconds.
34. $16 \times 99 =$ 1584
35. Which is smaller, $\frac{2}{3}$ or $\frac{5}{7}$? 2/3 (fraction)
36. $25^2 =$ 625

Number Sense 11820

MATH LEAGUE.ORG

Name Rooz

Grade 2nd

School manzahita

correct and incorrect must sum to last problem attempted

Place ID Sticker Inside This Box

Correct ____ Correct ____

Incorrect ____ Incorrect ____

Scorer Initials ____ Scorer Initials ____

1. $17 + 2017 =$ 2034
2. The hundreds digit of 31415 is 4
3. $14 + 41 =$ 55
4. $6 \times 34 =$ 204
5. The product of 9 and 6 is 15
6. $192 \div 6 =$ 32
7. Round 376 to the nearest ten. 380
8. The remainder of $52 \div 7$ is 3
9. $524 - 294 =$ 230
10. (estimate) $85 + 95 + 105 + 115 =$ 400
11. $11 \times 26 =$ 286
12. $9^2 =$ 81
13. $217 + 192 =$ 409
14. $50 \times 16 =$ 800
15. LXXXVIII in Arabic numerals is ???
16. $10 \times 71 =$ 710
17. The remainder of $245 \div 4$ is 1
18. $24 \times 15 =$ 360
19. $17 \times 23 =$ 391

16 1450

20. (estimate) $1111 + 222 + 33 + 4 =$ 1370
21. $\frac{3}{5} =$ 0.6 (decimal).
22. If 1 mile is equal to 5280 feet then 2 miles are equal to 10360 feet.
23. $25 \times 21 =$ 525
24. $5 \times 44 \div 20 =$ 21
25. The GCD of 16 and 12 is ____.
26. $\frac{1}{4} + \frac{1}{3} =$ ____ (fraction).
27. $40 \times 22 =$ ____.
28. $\frac{6}{9}$ in lowest terms is ____ (fraction).
29. $23^2 =$ ____.
30. (estimate) $19 \times 20 \times 21 =$ ____.
31. The perimeter of a square with a side of length 5 is ____.
32. $22 \times 28 =$ ____.
33. Eight minutes is ____ seconds.
34. $16 \times 99 =$ ____.
35. Which is smaller, $\frac{2}{3}$ or $\frac{5}{7}$? ____ (fraction)
36. $25^2 =$ ____.

Math Circle 6 Math League Continued 9-25-2024 1:55-300

Cyrus, Marc, and Dar

I continued with some of the basic numbers sense tricks that they can use for the their Math League qualification and also it is good for them to have in their Mental Math knowledge. I also gave them some quick quizes to practice doing the problems fast but accuratly.

I gave them first the cards that I had made based on the draft that I shared with them for finding remainders. I have found that if the cards are nicely made and laminated they pay more attention to them! We discussed the rules in particular for 7 and 11 and also for 8 that I also had a card made for them for divisibility by 8. Then I gave them the quiz that they did in 5 minutes. They finished before time. They got most of the answers but they both had problems with remainders for 7 and 11 that are the hardest.

Divisibility Check and Finding the Remainder
For 2) If the Number is Odd the remainder is 1. R=1
For 3) Subtract 1 or 2 to make the sum of the digits divisible by 3. For example: For 377 subtract 2 to get 375. R=2
For 4) Subtract 1, 2, or 3 to make the last 2 digits divisible by 4. (odd+2,6 - even+0,4,8) For example: For 51,939 subtract 3 to from 39 to get 36. R=3
For 5) Subtract 1,2,3, or 4 to make the last digit 0 or 5. e.g. for 69,427 subtract 2 to get 69,425. R=2
For 6) Subtract 1 to 5 to make the result even and divisible by 3. e.g. For 5263 subtract 1 to get 5262 that is even with sum of digits=15, R=1
For 7) Subtract 1 to 6 to make the result divisible by 7 (a-2b divisible by 7) e.g. for 14626 ignore 14 then for 625 subtract 2 to get 623 since 62-6=56=7x8 R=2
For 8) Subtract 1 to 7 to make the last 3 digits of the result divisible by 8 (e xx div by 8, oxx div by 4 not 8). e.g. for 35,671 subtract 7 from 671 to get 664 that is divisible by 8. R=7
For 9) Subtract 1 or 8 to make the sum of the digits divisible by 9. For example: For 36,782 subtract 8 to get 36,774 (sum of digits=27). R=8
For 11) Subtract 1 to 10 to make the alternating sum divisible by 11. e.g. for 35,237 3-5+2-3+7=4 subtract 4 to get 35,233 3-5+2-3+3=0. R=4
For 12) Subtract 1 to 11 so the result is divisible by 3 and 4. e.g.for 3615 subtract 3 to get 3612 sum of digits 12 and 12 is divisible by 4. R=3

Divisibility Rule for 8: Look at last 3 Digits

For 200,400,800 last 2 digits should be green.

For 100,300,500,700,900 last 2 digits should be red.

00		04		08
	12		16	
20		24		28
	32		36	
40		44		48
	52		56	
60		64		68
	72		76	
80		84		88
	92		96	

Name: Marc

the remainder of 362 ÷ 3 is 2

the remainder of 4,562 ÷ 5 is 2

The remainder of 36,323 ÷ 4 is 3

the remainder of 52,799 ÷ 10 is 9

the remainder of 391,631 ÷ 2 is 1

the remainder of 65,323 ÷ 6 is 1

the remainder of 655 ÷ 7 is 4

the remainder of 999,817 ÷ 9 is 7

the remainder of 71,410 ÷ 8 is 2

the remainder of 657,651 ÷ 12 is ~~11~~ 3

Name: Cyrus

the remainder of 362 ÷ 3 is 2

the remainder of 4,562 ÷ 5 is 2

The remainder of 36,323 ÷ 4 is 3

the remainder of 52,799 ÷ 10 is 9

the remainder of 391,631 ÷ 2 is 1

the remainder of 65,323 ÷ 6 is 1

the remainder of 655 ÷ 7 is 4

the remainder of 999,817 ÷ 9 is 7

the remainder of 71,410 ÷ 8 is 2

the remainder of 657,651 ÷ 12 is ~~11~~ 3

Next, I gave them the cards for the 20x20 Multiplication table that had the squares of the 2-digit numbers on the back and went over the fact that they needed to know these results pretty much like the 10x10 multiplication table. We went over the multiplication of one-digit numbers by numbers from 11to 20 and the squares of numbers from 11 to 19. I also went over the multiplication method for numbers between 11-19 that will help them find these values quickly. I went over the rules and we each asked a few of them.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200
11	22	33	44	55	66	77	88	99	110	**121**	132	143	154	165	176	187	198	209	220
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	**361**	380
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	**400**

1s Digit → 10s Value ↓	0	1	2	3	4	5	5	7	8	9
0	0	1	4	9	16	25	36	49	64	81
10	100	121	144	169	196	225	256	289	324	361
20	400	441	484	529	576	625	676	729	784	841
30	900	961	1024	1089	1156	1225	1296	1369	1444	1521
40	1600	1681	1764	1849	1936	2025	2116	2209	2304	2401
50	2500	2601	2704	2809	2916	3025	3136	3249	3364	3481
60	3600	3721	3844	3969	4096	4225	4356	4489	4624	4761
70	4900	5041	5184	5329	5476	5625	5776	5929	6084	6241
80	6400	6561	6724	6889	7056	7225	7396	7569	7744	7921
90	8100	8281	8464	8649	8836	9025	9216	9409	9604	9801

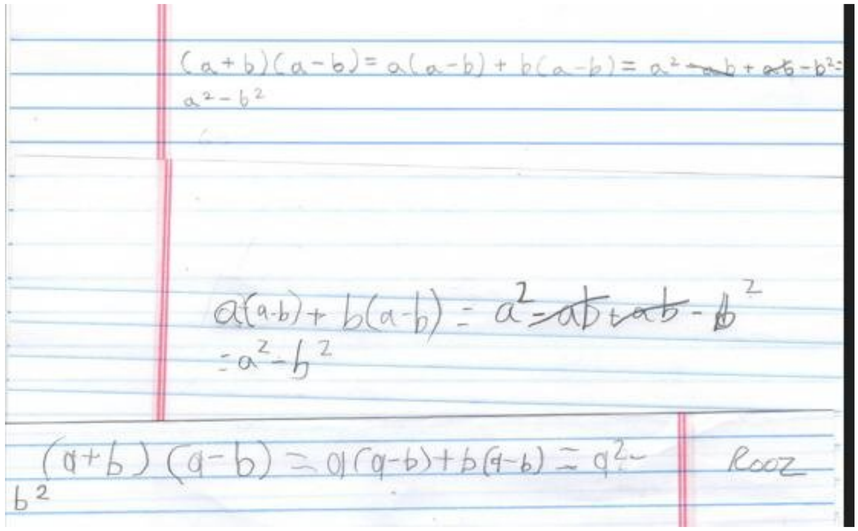
I also showed them how using the algebraic identity:

$$(a+b)(a-b)=a^2-b^2$$

$$(a+b)(a-b)+b^2=a^2$$

They can find the squares of two-digit numbers mentally much easier. I first made them do the algebra even though they were resisting. Rooz had also walked in, so I made him do it also partially.

For example to find 58^2you can think of it as (58-2)(58+2)+4=56x50+4=2,804 The trick is to make one of the numbers a multiple of 10 so they can do the multiplication easier in their head and add the square of the number they had to add to find the multiple of 10.

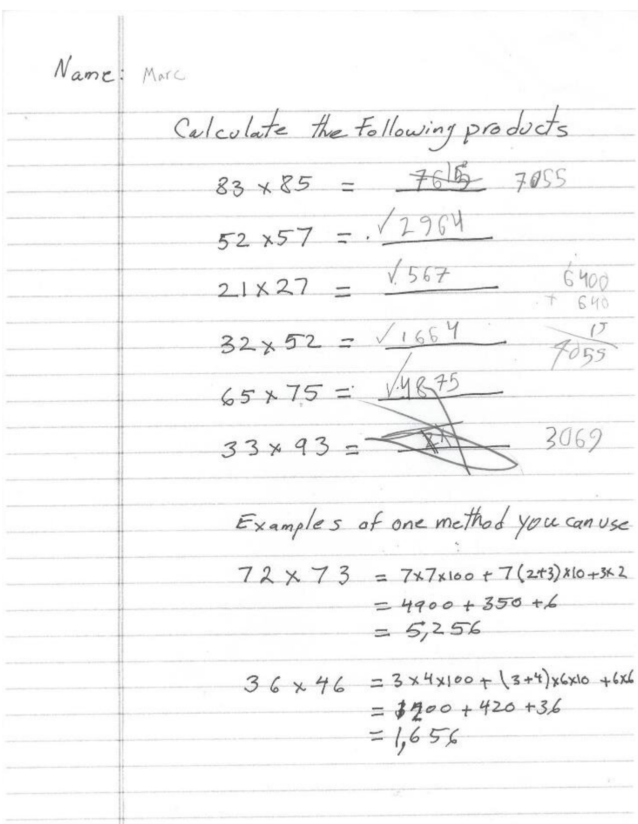


$(a+b)(a-b) = a(a-b) + b(a-b) = a^2 \cancel{-ab} + \cancel{ab} - b^2 = a^2 - b^2$

$a(a-b) + b(a-b) = a^2 \cancel{-ab} + \cancel{ab} - b^2 = a^2 - b^2$

$(a+b)(a-b) = a(a-b) + b(a-b) = a^2 - b^2$ Rooz

I also talked about the double-digit mental multiplication when the ten-digits are the same or the ones-digits are the same and gave them a short quiz to do. Marc got a little frustrated at the end as this method needs a bit more practice.



Name: Marc

Calculate the following products

$83 \times 85 =$ ~~7615~~ 7055

$52 \times 57 =$ √2964

$21 \times 27 =$ √567   6400 + 640

$32 \times 52 =$ √1664   ~~7055~~

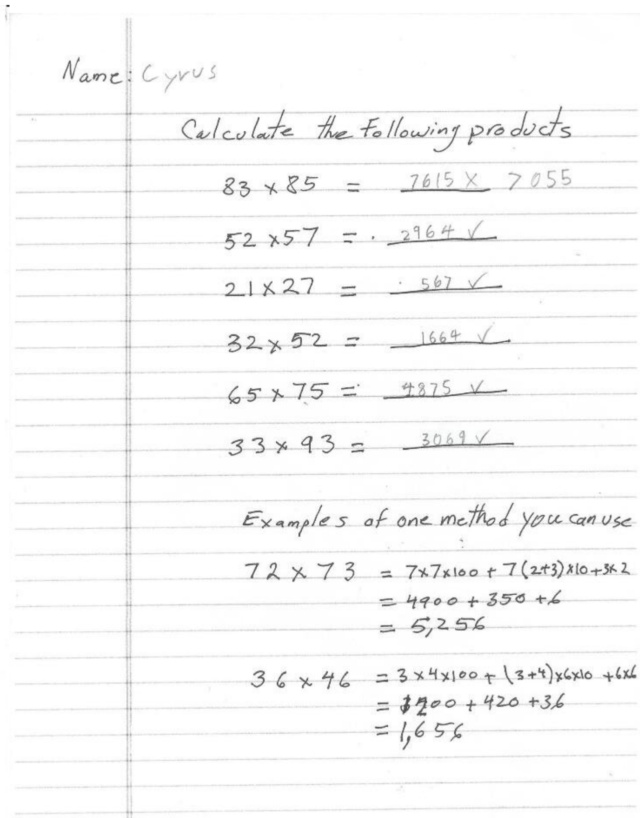
$65 \times 75 =$ √4875

$33 \times 93 =$ 3069

Examples of one method you can use

$72 \times 73 = 7 \times 7 \times 100 + 7(2+3) \times 10 + 3 \times 2$
$= 4900 + 350 + 6$
$= 5{,}256$

$36 \times 46 = 3 \times 4 \times 100 + (3+4) \times 6 \times 10 + 6 \times 6$
$= 1200 + 420 + 36$
$= 1{,}656$



Name: Cyrus

Calculate the following products

$83 \times 85 =$ 7615 X 7055

$52 \times 57 =$ 2964 √

$21 \times 27 =$ 567 √

$32 \times 52 =$ 1664 √

$65 \times 75 =$ 4875 √

$33 \times 93 =$ 3069 √

Examples of one method you can use

$72 \times 73 = 7 \times 7 \times 100 + 7(2+3) \times 10 + 3 \times 2$
$= 4900 + 350 + 6$
$= 5{,}256$

$36 \times 46 = 3 \times 4 \times 100 + (3+4) \times 6 \times 10 + 6 \times 6$
$= 1200 + 420 + 36$
$= 1{,}656$

This was a good session and these mental tricks as well as some others that I will teach them should help them in the Math League if they get selected.

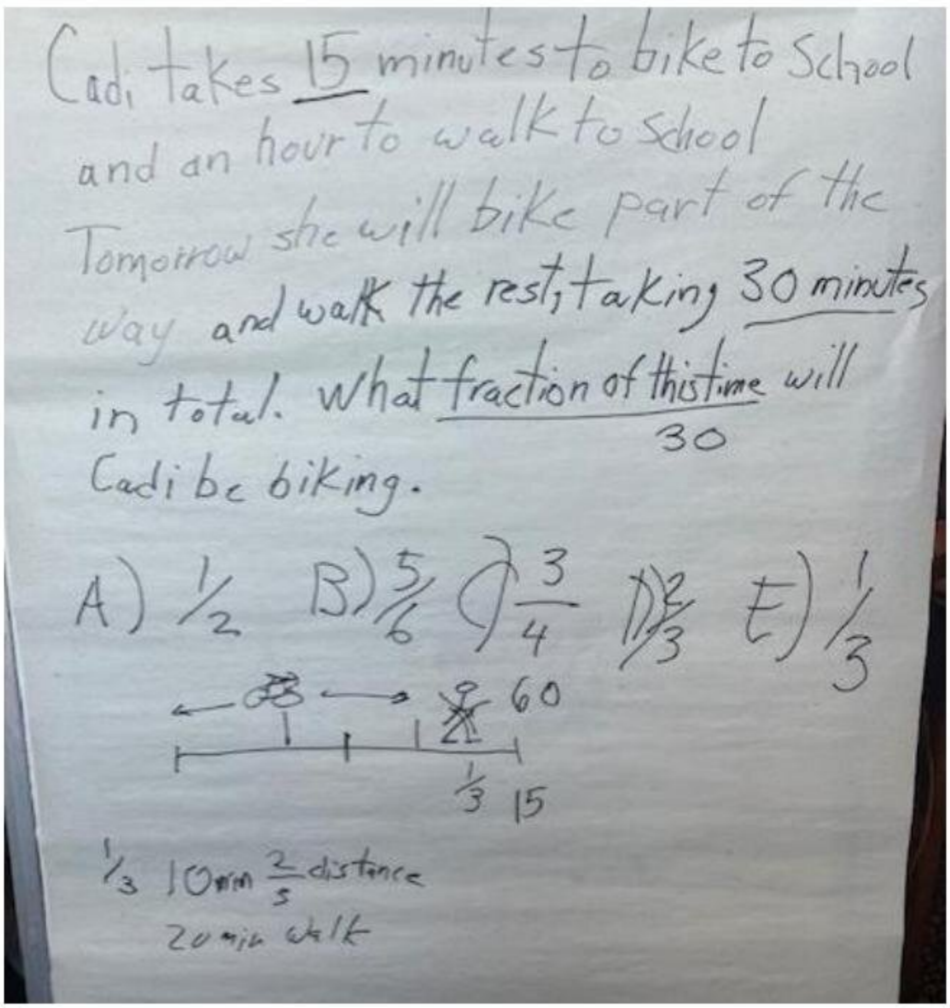
Math Circle 7 Math League Continued 10-16-2024 1:55-3:00

Math Circle CMR 7 Math League Continued 10-16-2024 1:55-3:00

Cyrus, Marc, Rooz, and Dar

There was no MC last week because of holiday. We started with a new format with Cyrus, Marc and Rooz and will continue as long as that works. If need be I will have a separate session for Rooz sometimes. Based on the results of this session I think we should be able to continue with it. I will address some of the tactics that they could use for math league competion and beyond but with an emphesis on finding patterns and generalizing from it. Also I will try to get them to discuss how they would solve different problems. We will work on some of the tricks and tip from the Math league book that Laura shared with me.

I First choose a sprint question that I had trouble with and then chose 2 topics: Squares up to 100 and Cubes up to 20.



I started by telling them that they were a team and since Rooz was younger they were to help him along and not give him or each other any trouble. I also told them about the different parts of the test a bit and about the Sprint part that was either a multiple choice or required a written answer with 30 questions.

I showed them the problem and read it for them and asked them to think about it and tell me what they thought the right answer may be. Both Marc and Cyrus after a bit of back and forth chose 2/3, Rooz chose the right answer 1/3 (I had come up with 2/3 myself when I first looked at the problem. Without telling them what the correct answer was I engaged them in and on how they could go about the finding the correct solution. By starting with ½ and if ½ did not work should the answer be more than ½ or less than ½. Some how Marc and Cyrus had not paid attention that ½ had to be applied to the 30 minutes and not the 15 minutes. After some back and forth I did point out that if ½ to 30 was biking then Cadi would have covered the whole distance on the bike, so the answer had to be 1/3. I also drew a picture, and I said they should always try to draw on paper if possible and if not in their head. Cyrus and Marc did give a dirty look to Rooz as they were adamant that their answer was correct. I have to be careful that they really become a team;=>

Next, I told them that they were going to learn a method to find squares of numbers from 50 to 60 and use this method to find all the squares of numbers from 10 to 100.

I mentioned again that they would be using the following algebraic identity:

$$(a+b)(a-b)=a^2-b^2$$

$$(a+b)(a-b)+b^2=a^2$$

Rooz wanted to do his method and was making faces, but I got him to use this method and once he saw the pattern he was off and running. I gave them the card that I had made for them, and they got going. Marc finished first and Cyrus was moving much faster, and he said so and Rooz finished right after Cyrus and even followed the pattern for the last row.

x	a	(50+a)(50-a)	Base	+	a^2	x^2
50	0	50x50=2,500	2,500	+	0	2,500
51	1			+		
52	2			+		
53	3			+		
54	4			+		
55	5			+		
56	6			+		
57	7			+		
58	8			+		
59	9			+		
60	10			+		

x	a	(50+a)(50-a)	Base	+	a^2	x^2
50	0	50x50=2,500	2,500	+	0	2,500
51	1	50×52=2600	2600	+	1	2601
52	2	50×54=2700	2700	+	4	2704
53	3	50×56=2800	2800	+	9	2809
54	4	50×58=2900	2900	+	16	2916
55	5	50×60=3000	3000	+	25	3025
56	6	50×62=3100	3100	+	36	3136
57	7	50×64=3200	3200	+	49	3249
58	8	50×66=3300	3300	+	64	3364
59	9	50×68=3400	3400	+	81	3481
60	10	60×60=3600	3600	+	0	3600

a	a^2	$a^3=a.a^2$	Delta 1	Delta 2	Delat 3	Delta 4
0	0	0x0=0				
1	1	1	1-0=1			
2	4	8	7	6		
3	9	27	19	12	6	
4	16	64	37	18	6	0
5	25	125	61	24	6	0
6	36	216	91	30	6	0
7	49	343	127	36	6	0
8	64	512	169	42	6	0
9	81	729	217	48	6	0
10	100	1000	271	54	6	0

x	a	(50+a)(50-a)	Base	+	a^2	x^2
50	0	50x50=2,500	2,500	+	0	2,500
51	1	50x52=2,600	2,600	+	1	2,601
52	2	50x54=2,700	2,700	+	4	2704
53	3	50x56=2,800	2,800	+	9	2809
54	4	50x58=2,900	2,900	+	16	2,916
55	5	50x60=3,000	3,000	+	25	3,025
56	6	50x62=3,100	3,100	+	36	3,136
57	7	50x64=3,200	3,200	+	49	3,249
58	8	50x66=3,300	3,300	+	64	3,364
59	9	50x68=3,400	3,400	+	81	3,481
60	10	50x70=3,500	3,500	+	100	3,600

a	a^2	$a^3=a.a^2$	Delta 1	Delta 2	Delat 3	Delta 4
0	0	0x0=0				
1	1	1	1-0=1			
2	4	8	7	6		
3	9	27	19	12	6	
4	16	64	37	18	6	0
5	25	125	61	24	6	0
6	36	216	91	30	6	0
7	49	329	113	36	6	0
8	64	512	197	42	6	0
9	81	729	217	48	6	0
10	100	1,006	271	54	6	0

Once they were done, we discussed the fact that they could now easily remember (or recompute in their head) any of these squares as they could start with the base of 2,500 and add as many 100s as the 1s digit then add the 1s digit squared to it so for 58^2

$$58^2 \Rightarrow 2500 \Rightarrow 3300 \Rightarrow 3364$$

Next, I gave them the revised squares table below and encouraged them to find a similar pattern first for 20s then for 30s, 40s etc. They quickly concluded that they could do the same except now instead of adding 100s(50x2) they needed to add 40(20x2) for 20s and 60(30x2) for 30s etc. I asked them to work on this and see how many of these squares they could do in their head by next week. We tried a number of the squares and Rooz came up faster a few times but then confessed that he was looking at the card;=> I think they were impressed as much with the method as much as I was with coming up with it. May be I get the Field Medal for it;=>

1s Digit → 10s Value ↓	0	1	2	3	4	5	6	7	8	9	10
0	0	1	4	9	16	25	36	49	64	81	100
10	100	121	144	169	196	225	256	289	324	361	400
20	400	441	484	529	576	625	676	729	784	841	900
30	900	961	1,024	1,089	1,156	1,225	1,296	1,369	1,444	1,521	1,600
40	1,600	1,681	1,764	1,849	1,936	2,025	2,116	2,209	2,304	2,401	2,500
50	2,500	2,601	2,704	2,809	2,916	3,025	3,136	3,249	3,364	3,481	3,600
60	3,600	3,721	3,844	3,969	4,096	4,225	4,356	4,489	4,624	4,761	4,900
70	4,900	5,041	5,184	5,329	5,476	5,625	5,776	5,929	6,084	6,241	6,400
80	6,400	6,561	6,724	6,889	7,056	7,225	7,396	7,569	7,744	7,921	8,100
90	8,100	8,281	8,464	8,649	8,836	9,025	9,216	9,409	9,604	9,801	10,000
100	10,000	10,201	10,404	10,609	10,816	11,025	11,236	11,449	11,664	11,881	12,100

Next, as Cyrus was asking me what was the discovery that I had made between the link between what he did with 3ed degree equation – pascal's triangle and cubes that I mentioned to him and Rooz briefly on Sunday I started them on a method to find cubes for numbers between 1 and 10 (next week we will do the same exercise with numbers between 10 and 20.

I gave them the template that I had prepared for them and they got started:

a	a^2	$a^3=a.a^2$	Delta 1	Delta 2	Delat 3	Delta 4
0	0	0x0=0				
1	1	1	1-0=1			
2						
3						
4						
5						
6						
7						
8						
9						
10						

They kept asking what the deltas were, and I told them that they would find out. Mark was proud to have found that I misspelled Delat3 instead of Delta 3 and of course I said, as usual, I did it on purpose to see if anybody catches it;=> Marc was running ahead a usual but Cyrus and Rooz were keeping up. Both Marc and Rooz had to restart Delta 2 s they went too fast, but they all were impressed that delta 3 became 6 and Delat 4 became 0. They wanted to find some pattern for Delat 1 like it were all primes, but they found that ninety-one was not prime.

Next, I asked them to see what the difference was for the consecutive squares. Cyrus was convinced that this was related to a discovery he had made a few years back about the sum of consecutive step squads being a square. (I have to think about this one) They first said that these will be the primes and quickly everyone got it that it was just the consecutive odd numbers and this is related to the sum of consecutive odd numbers always being a square number. Indeed:

$$1+3+5+\ldots+n \text{ is always equal to } ((n+1)/2)^2$$

I will have to review this with them next time also.

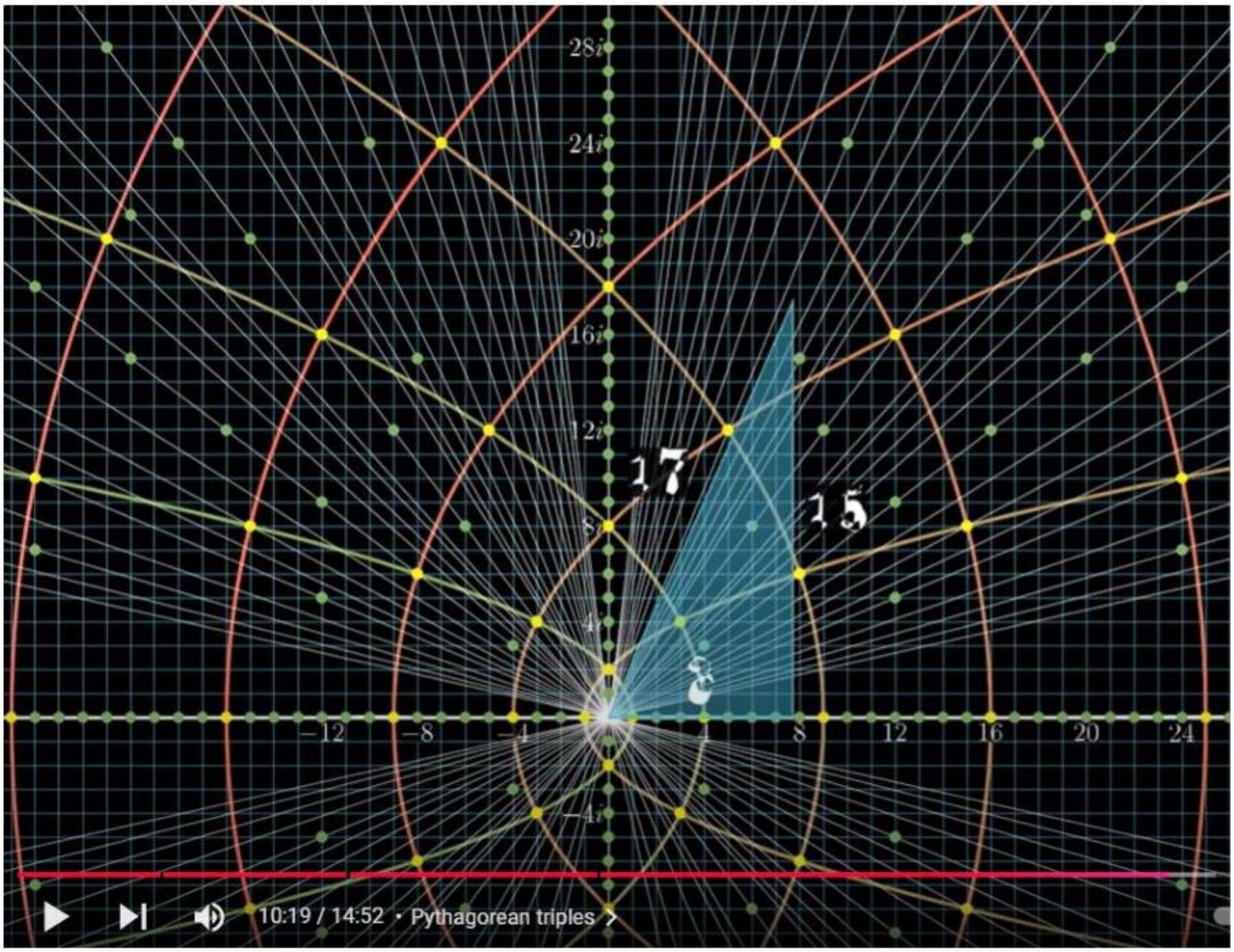
It was our best Math Circle ever and both Marc and Cyrus hugged me and said this was the best math Circle ever. In addition, Rooz established himself as a legitimate member of this new Math Circle. We had so many more discussions than usual and they did not even ask for the time once during the session.

Math Circle 8 Pythagorean Triples 10-23-2024 12:15-1:15

Math Circle CMR 8 Pythagorean Triples 10-23-2024 12:15-1:15

Cyrus, Marc, Rooz, and Dar

I was going to continue with the Math League stuff and in particular do the Cubes from 10-20 and have them workout why the successive diferrence go to zero. But I saw a video about Pythagorean Triples that was amazing and instead we worked on that. This video was from 3 Blue 1Brown and was called: [All possible Pythagorean triples, visualized] and it is a remarkable piece of Mathematics.



I prepared a worksheet for them and discussesd the fact that they could use the following formulas to find Pythagorean Triples

$$(a+b)(a-b)=a^2-b^2$$

$$(a+b)(a-b)+b^2=a^2$$

They undrestood quickly that if one side is an odd number they could find two consecutive numbers that added up to the square of that number. They also came up with some odd numbers of their own and found the other two numbers in the triple.

Next I used the method that Grant had discussed in the video but did not use complex numbers and came up with a version that just has real integers. In short if we have any two numbers x>y then the following is a Pythagorean triple:

$$x^2-y^2,\ 2xy,\ x^2+y^2$$

I had shown the video to Rooz and Cyrus once and they saw the magic but I am not sure they undrestood it all (I did not undrestand it fully myself and had to watch it again and still they are some stuff that I have to think about more). Never the less they used the method and found that all the triples that they had found can now be found using two numbers x and y for example 2, 1 gives the triple 3, 4,5. They had some trouble with the ones they had come up with but I helped them some and finally Cyrus found that a=2x-1. So x=(a+1) and in our case here y=x+1.

I had them pick a couple or random x and y and see that they always got a Pythagorian Triple with any two integers x and y (this is truly remarkable).

At this point Mihai and Hugo came and I ran the video for everybody and I think everybody connected with it. Even Rooz that has not worked with imaginary numbers was impressed and I could see the gleam in his eyes.

This was another great Math Circle;=>

$a=x^2-y^2$	$b=2xy$	$c=x^2+y^2$	$a^2=c^2-b^2=(b+c)(b-c)$		x	y
3	4	5	$3^2=5^2-4^2=(5+4)(5-4)=9$		2	1
5	12	13	$5^2=12+13\times1=25$		3	2
7	24	25	$7^2=24+25\times1=49$		4	3
9	40	41	$9^2=40+41\times1=81$		5	4
11	60	61	$11^2=60+61\times1=121$		6	5
13	84	85	$13^2=84+85\times1=169$		7	6
15	112	113	$15^2=112+113\times1=225$		8	7
31	480	481	$31^2=480+481\times1=961$		16	15
101	5100	5101	$101^2=5100+5101\times1=10201$		51	50

$a=x^2-y^2$	$b=2xy$	$c=x^2+y^2$	$a^2=c^2-b^2=(b+c)(b-c)$		x	y
3	4	5	$3^2=5^2-4^2=(5+4)(5-4)=9$		2	1
5	12	13	$5^2=12+13\times1=25$		3	2
7	24	25	$7^2=24+25\times1=49$		4	3
9	40	41	$9^2=40+41\times1=81$		5	4
11	60	61	$11^2=60+61\times1=121$		6	5
13	84	85	$13^2=84+85=169$		7	6
15	112	113	$15^2=112+113\times1=225$		8	7
91	4,440	4,441	$91^2=4,440+4,441\times1=8281$		46	45
89	3,160	3,161	$89^2=3160+3,161\times1=7,921$		45	44

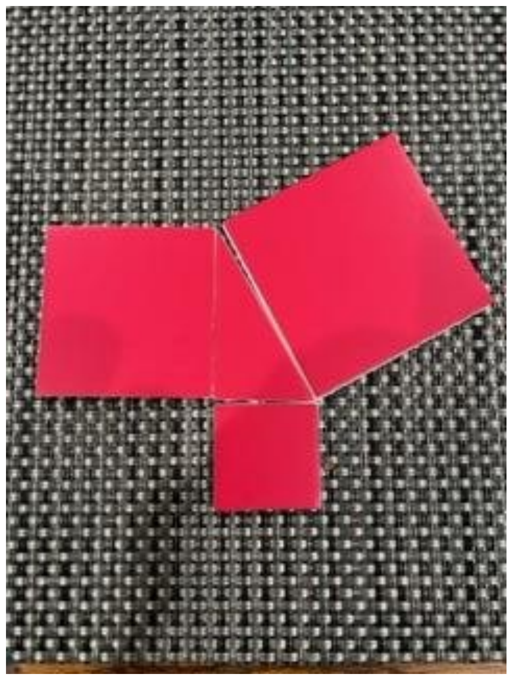
Math Circle 9 Pythagorean Triples 10-30-2024 12:30-1:35

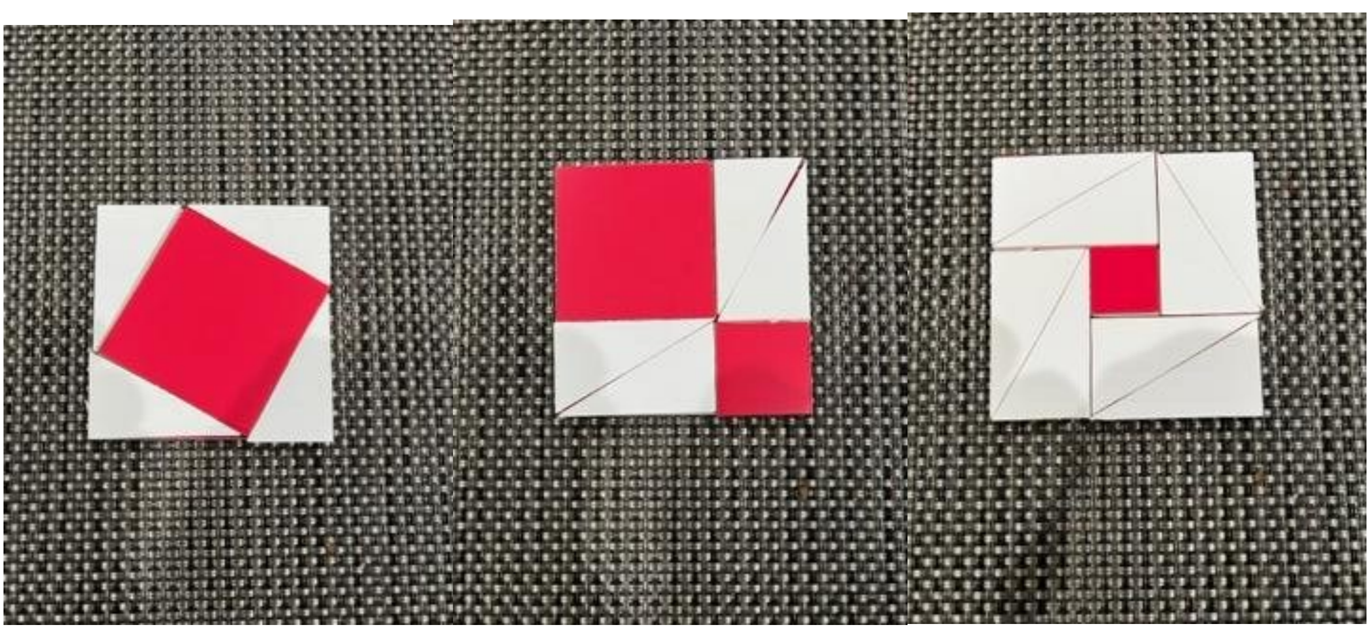
Cyrus, Marc, Rooz, and Dar

We started with the visual proof of the Pythagorean Theorem that Cyrus and Marc had seen two years back but was new for Rooz. They each demonstrated it with the material and also we talked about how the the following identeties can be seen as a by product of this proof: Cyrus also mentioned that we had done the proof for a^2-b^2= (a+b)(a-b)but I did not pursue that as I wanted to show them some of the other proofs.

$$(a+b)^2 = a^2+b^2+2ab.$$

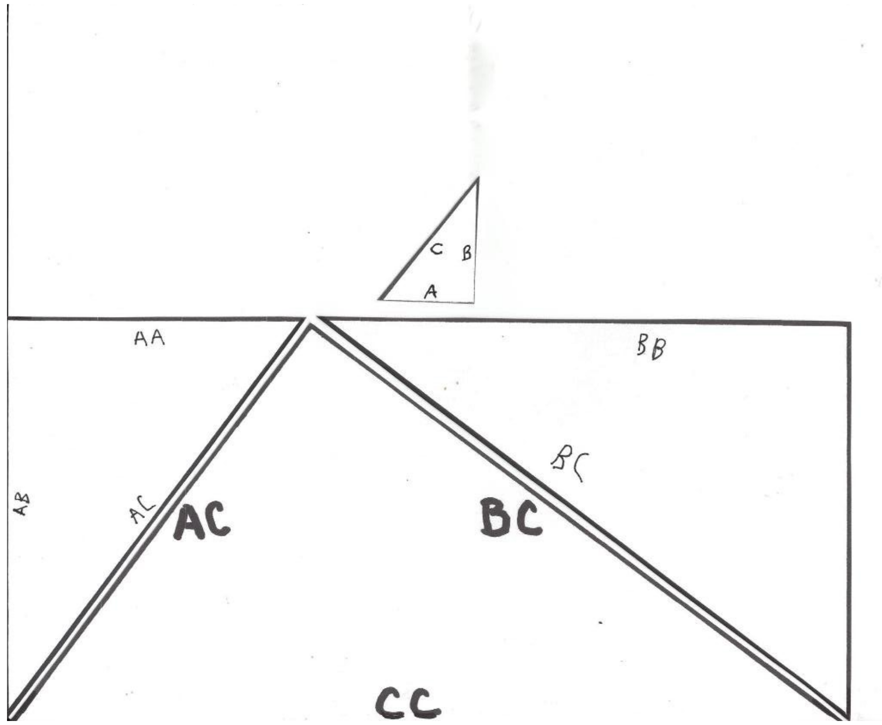
$$(a-b)^2 = a^2+b^2-2ab.$$

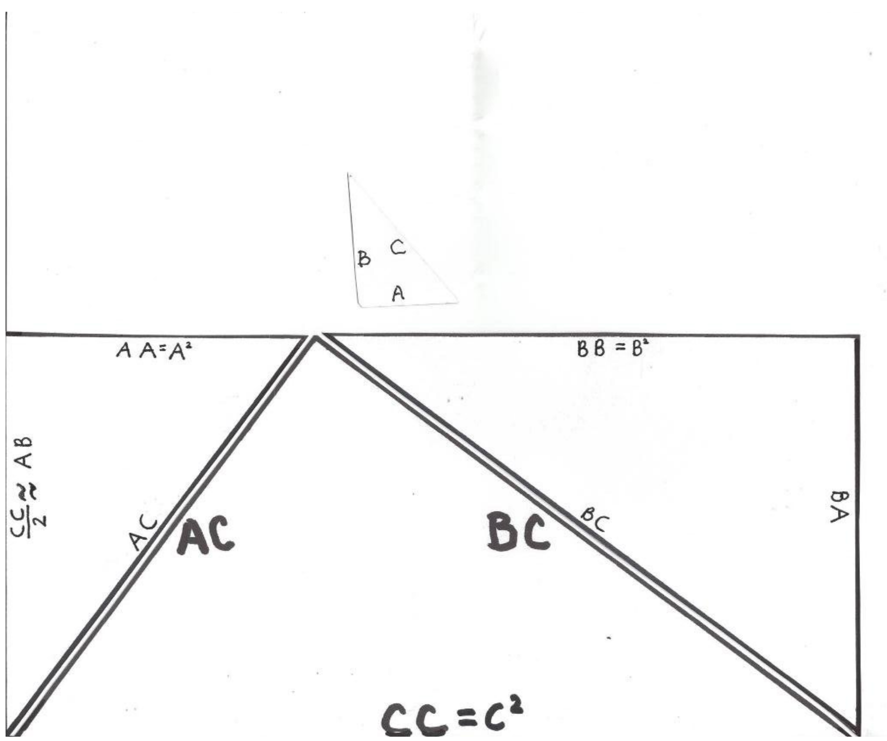




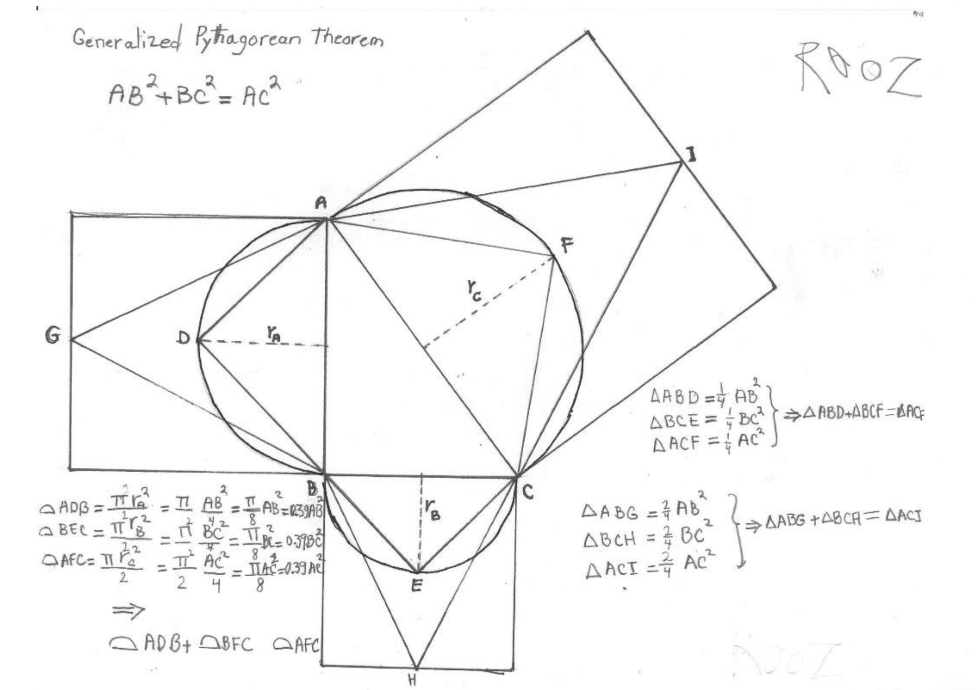
They also did some of the other proofs, in particular the one below that came from a Mathaloger youtube video. [**Pythagoras twisted squares: Why did they not teach you any of this in school?**]

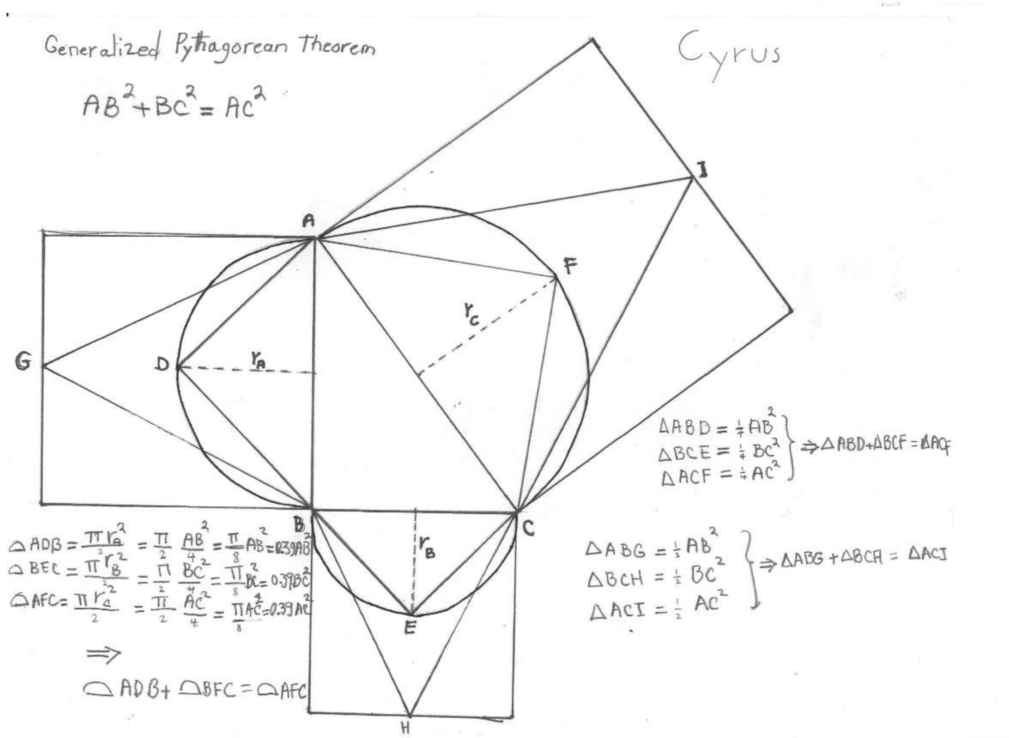
They had to mark the two smaller trianges that were the scaled version of the very small triangle by A and B to see that the opposite sides of the rectagle showed AA+BB=CC or $A^2+B^2=C^2$. I helped them a bit but they all got there.





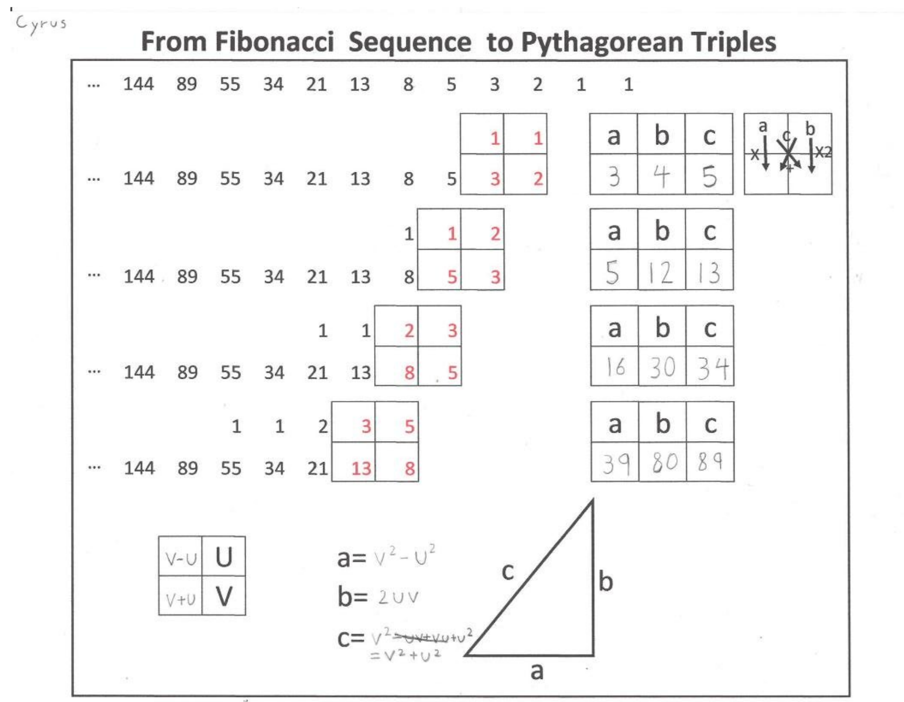
Next they worked on the worksheet that I had prepared for them from the same Mathologer video that showed that instead of squares they can use any triangle or even a half circles or any other shape they would have an equivalent proof as long as these shapes are similar and built on the 3 sides of the right triangle. They had a bit of trouble with half circles as I had not left them enough space for them to put in the coefficients but eventually they did it with help. [This video would be great to watch] also.

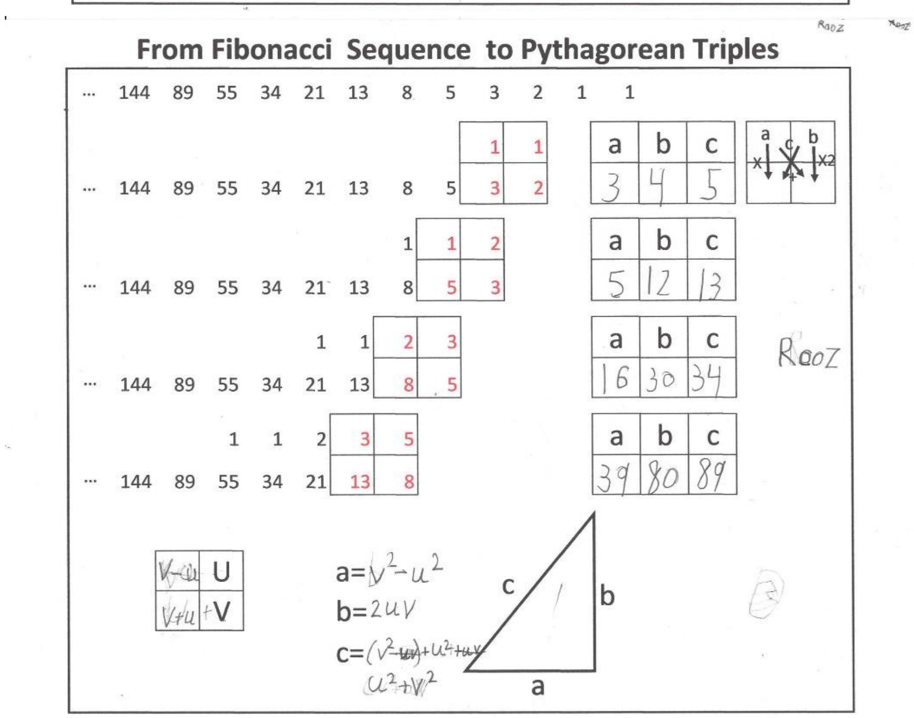




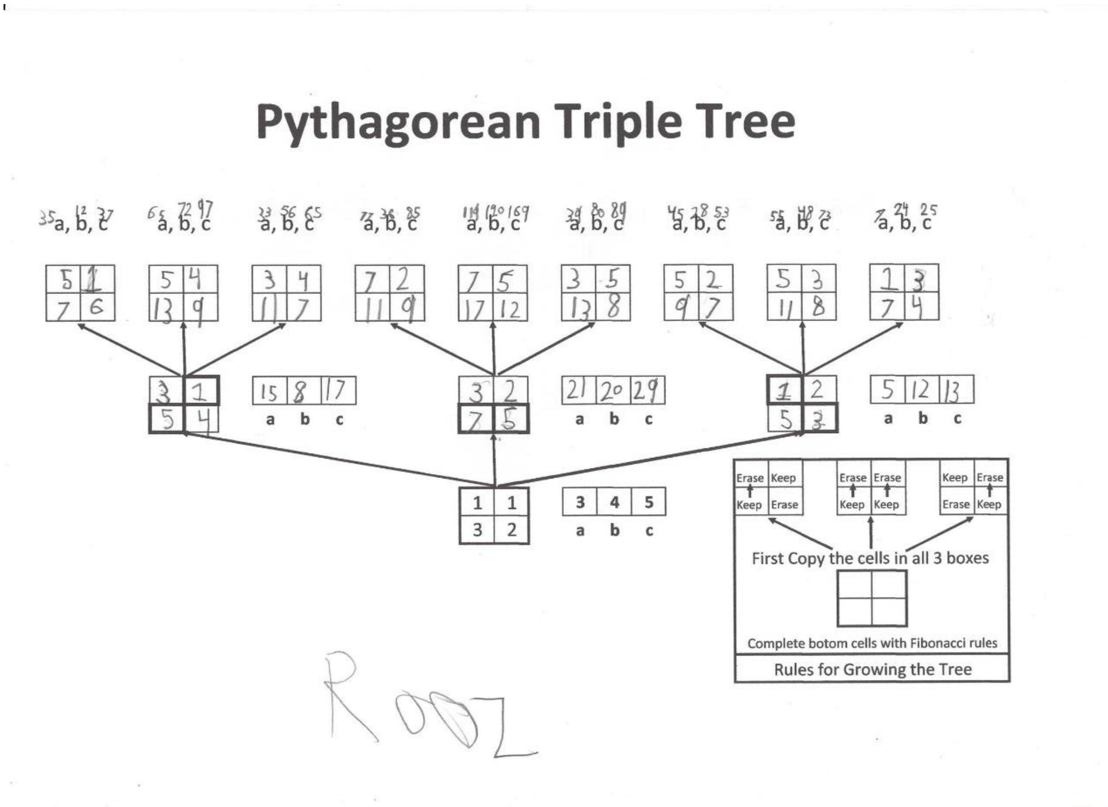
Next, I gave them worksheets that I had prepared based on a related Mathologer video: [Fibonacci = Pythagoras: Help save a beautiful discovery from oblivion].

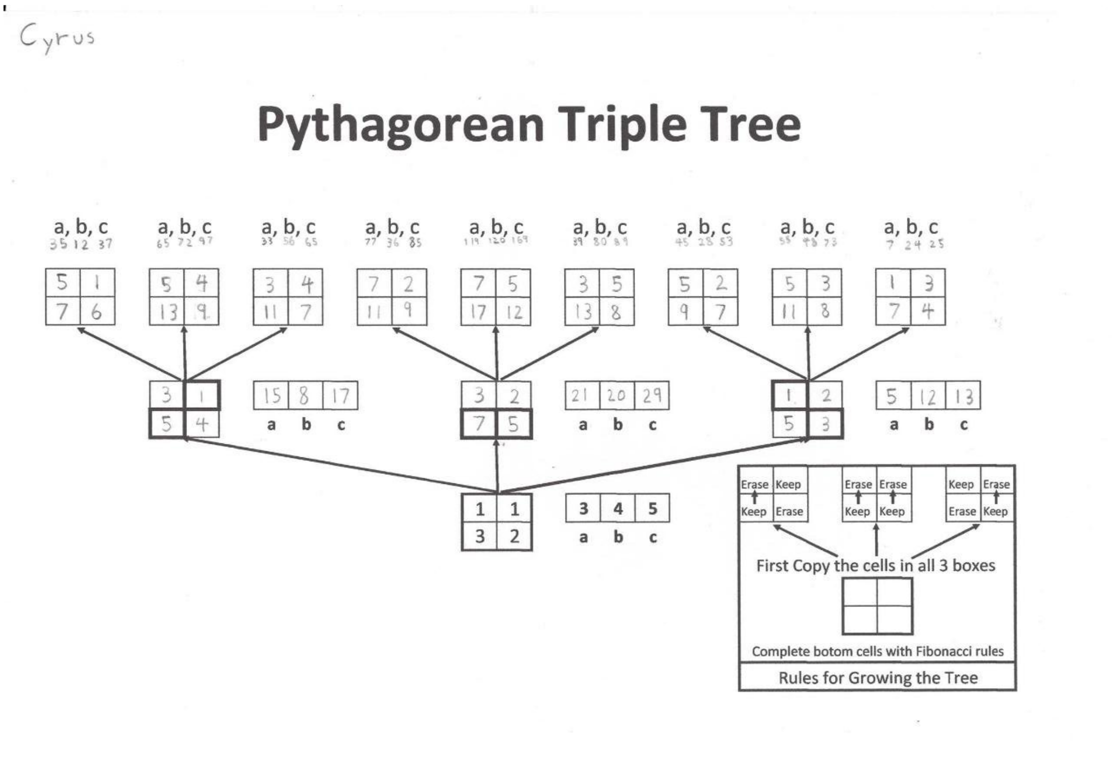
They had no trouble with the worksheets and even finding the formulas for a, b, and c the sides and hypothenus and the other sides of the right trinagle base on U and V from the last box. Rooz needed a little help but he is doing great and even completed the other work on the sheet first. They saw that this last relation was the same as they had seen in the previous session $a=V^2-V^2$, $b=2UV$, $c=V^2+U^2$



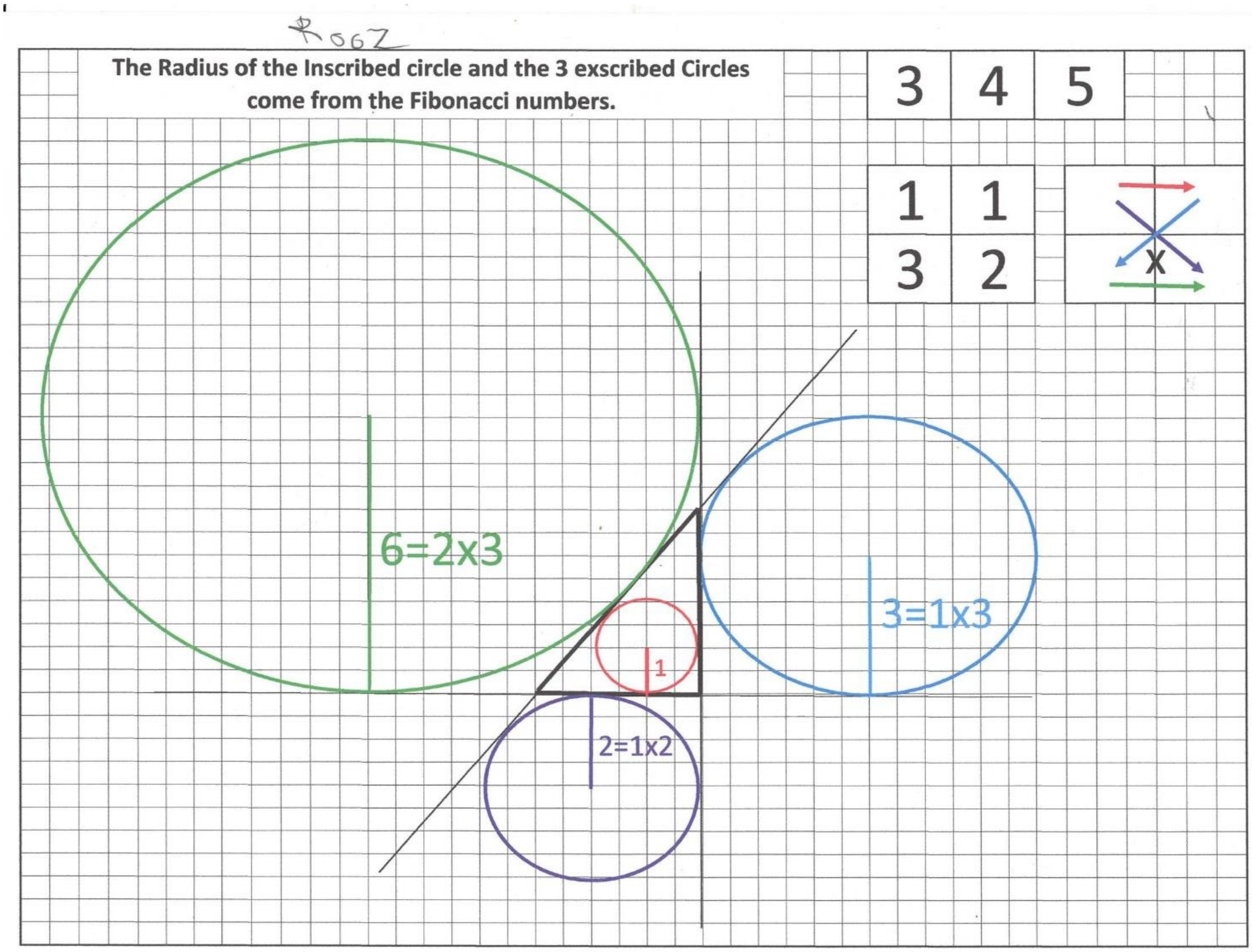


Next they worked on the worksheet for the Pythagorean Triple Tree and we discussed that these boxes will produce all the Primitive Pythagorean Triples exactly once! And in addition, the remarkable fact that the numbers in the boxes seen as fractions will produce all the simplefied fractions starting with ½ and 1/3!





We also discussed how the radius of inscribes (in circle) and the 3 excribed (out circles) circles could also be found from box for the 3, 4, 5 triangle and all other right triangles.



This was a great session and they enjoyed it as much as I had enjoyed learning about the Pythagorean Triple Tree and the other facts from the Mathaloger Videos and preparing the worksheets for them.

Math Circle 10 Fibonacci numbers – Proof by Induction 11-06-2024 1:55 3:25

Cyrus, Marc, Rooz, and Dar

We started with the Fibanacci sequence. I discussed many of the the facts about the Fibonacci sequence that they would later watch in the video: **The magic of Fibonacci numbers | Arthur Benjamin | TED.** Including the fact that the sum of the consequetive Fibonacci numbers become a Fibonacci number (This we actullay did later) as well as 5 is the 5th Fibonacci number and the the 12th Fibonacci number is 144 and this does not happen again Cyrus brought this up as we had discussed it before and we wondered if there is a Fibonacci number that is the cube of it's order number (we have to research this).

Then I gave them The work sheet that I had prepared for them to see that the sum of squares of the Fibanacci Numbers will be the product of the last number in the sequence and the one after that. $F_1^2+F_2^2+F_3^2+\ldots+F_n^2=F_nxF_{n+1}$. They did this pretty quickly and I had them see and write down each product and finally come up with the formula fo n. They did this by comparing it to the rectangles that are produced by the squares of the Fibonacci numbers on their worksheet. Rooz needed a bit more help with the case of n as this is getting too abstract for him but Cyrus and Marc saw it pretty quickly once they had written all the other numerical products.

Next I Explained what the Proof by Induction was and how it worked. Cyrus asked a lot of questions that I am sure Marc and Rooz also had. Finally I just told them to trust me and it Proof by Induction was sort of magical and even seemed like cheating the way it works. Finally I had them do the last step and see that for the case of n+1 the identety will follow because $F_nxF_{n+1++}F^2_{n+1}= F_{n+1}(F_n + F_{n+1})$ and the last tem is equal to F_{n+2} by the basic property of Fibonacci numbers. This needed more explanation on how the factoring worked but I think all 3 of them in particular Cyrus and Marc had an Aha moment when they saw how easily the formula for n+1 jumps out the formula for n.

Next, they watched the video: : **The magic of Fibonacci numbers | Arthur Benjamin | TED** followed by the video: **The Golden Ratio: Is It Myth or Math?** That they enjoyed.

This was a good session I did not get to do many of the things I had prepared for them but I think they all got a good introduction to how the Proof by Induction works. This is a very advanced topic so we spent a lot of time talking about it. Cyrus will not let go until he gets it! Even then at some point I had to ask him to just trust me and he did.

I will do more of the Fibonacci and Proof by Induction for them next session to make sure they really get it.

Cyrus

Fibonacci Sequence Properties

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	...
F_n^2	1	1	4	9	25	64	169	441	1,156	3,025	7,921	20,736	54,289	142,129	372,100	974,169	...

1+1 = 2 = 1 × 2 = $F_2 \times F_3$

1+1+4 = 6 = 2 × 3 = $F_3 \times F_4$

1+1+4+9 = 15 = 3 × 5 = $F_4 \times F_5$

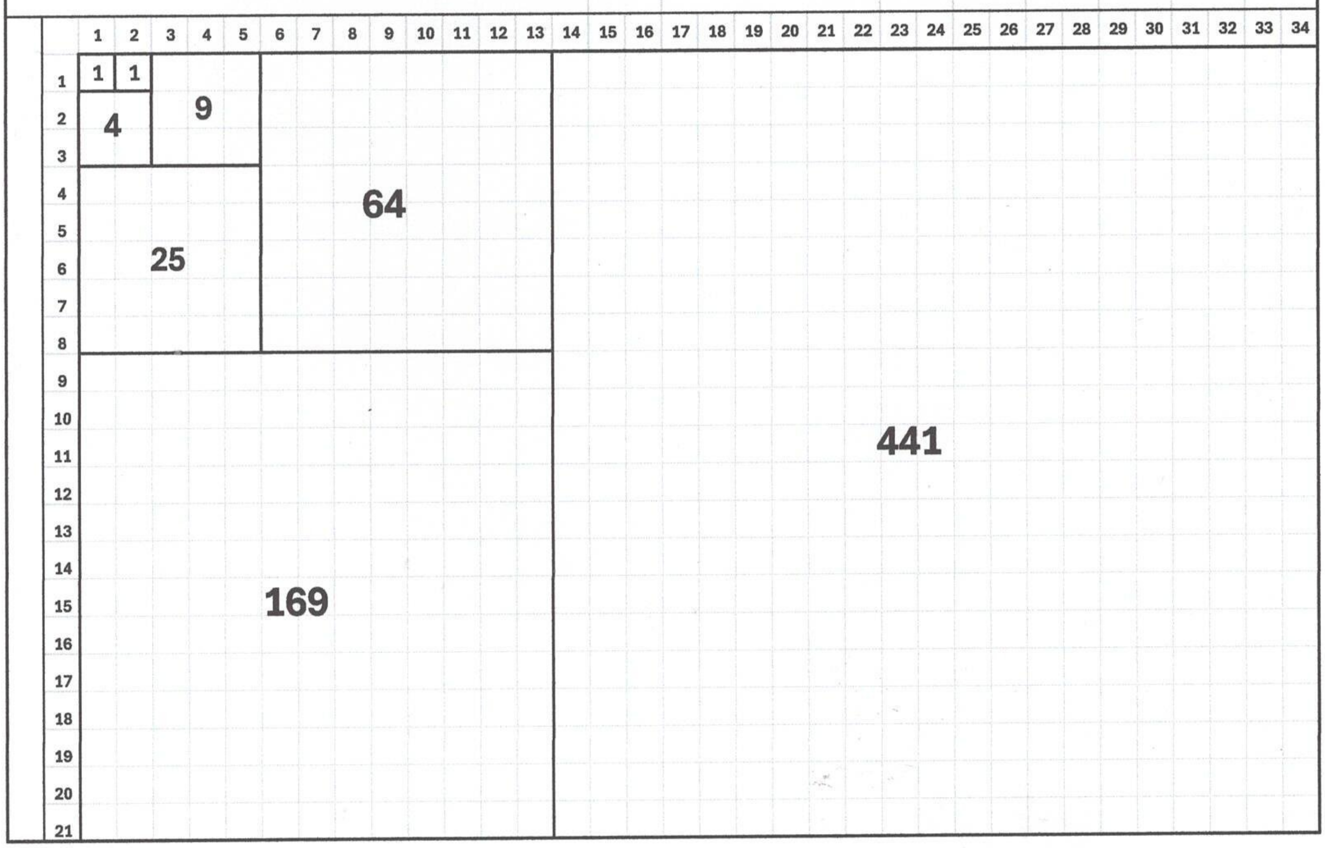
1+1+4+9+25 = 40 = 5 × 8 = $F_5 \times F_6$

1+1+4+9+25+64 = 104 = 8 × 13 = $F_6 \times F_7$

1+1+4+9+25+64+169 = 273 = 13 × 21 = $F_7 \times F_8$

1+1+4+9+25+64+169+441 = 714 = 21 × 34 = $F_8 \times F_9$

$$\sum_{i=1}^{n} F_i^2 = F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n \times F_{n+1}$$



Fibonacci Sequence Properties

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	...
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1+1 = 2 = 1 × 2 = $F_2 \times F_3$

1+1+4 = 6 = 3 × 2 = $F_3 \times F_4$

1+1+4+9 = 15 = 3 × 5 = $F_4 \times F_5$

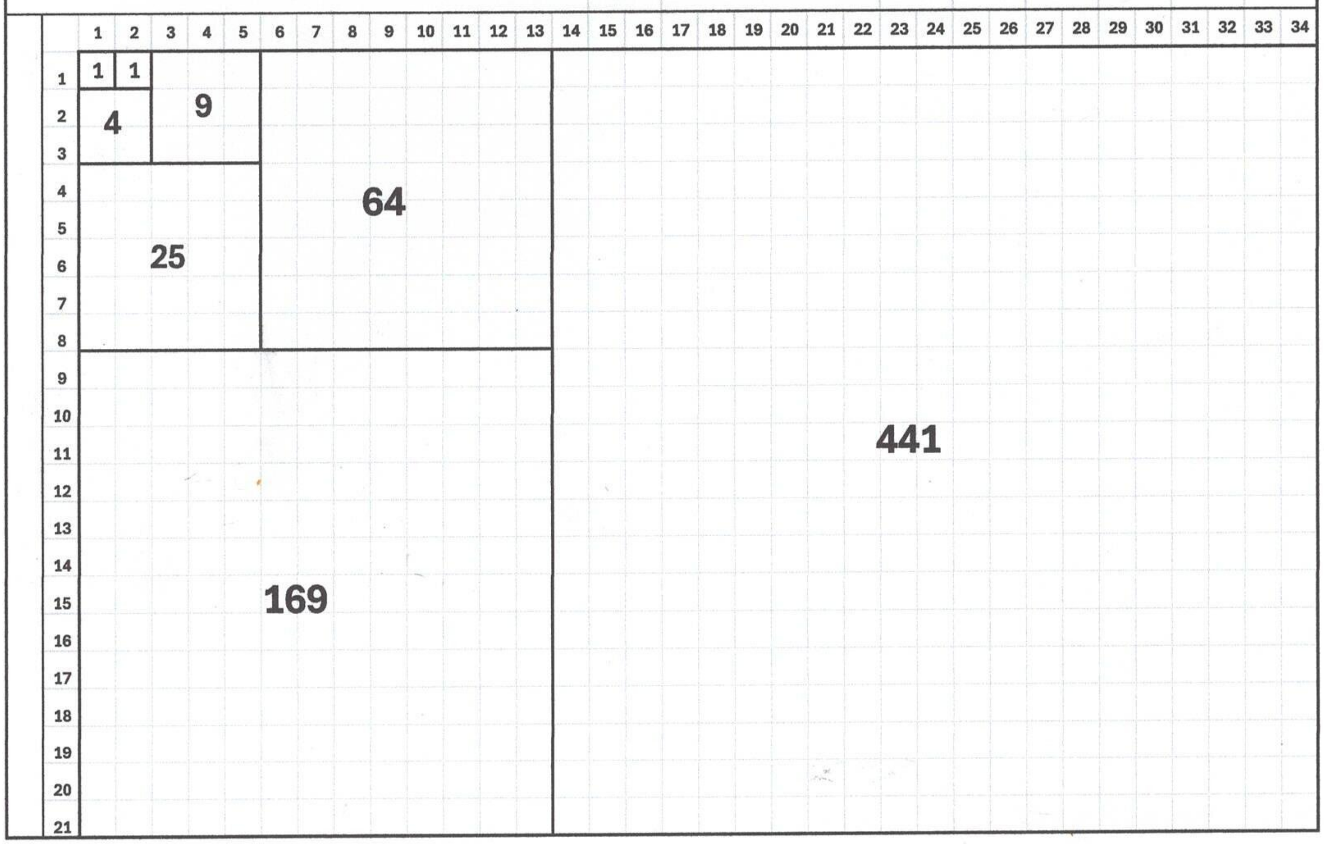
1+1+4+9+25 = 40 = 5 × 8 = $F_5 \times F_6$

1+1+4+9+25+64 = 104 = 8 × 13 = $F_6 \times F_7$

1+1+4+9+25+64+169 = 273 = 13 × 21 = $F_7 \times F_8$

1+1+4+9+25+64+169+441 = 714 = 21 × 34 = $F_8 \times F_9$

$$\sum_{i=1}^{n} F_i^2 = F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n \times F_{n+1}$$



Raoz

Proof by Induction

Prove That:

$$\sum_{i=1}^{n} F_i^2 = F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n \times F_{n+1}$$

Proof by Induction:

1. Check that it is true for n=1

2. Assume that it is true for n:

$$F1^2 + F2^2 + F3^2 + \ldots + Fn^2 = F_n X F_n + 1$$

3. Show that it is true for n+1

$$F1^2 + F2^2 + F3^2 + \ldots + Fn^2 + F_{n+1}^2 = F_{n+1} X F_{n+2}$$

Use the fact that $F_{n+2} = F_n + F_{n+1}$ (Definition of Fibonacci Sequence)

$n = 1 = 1 \times 1$ ✓ $F_1^2 + F_2^2 \; F_3^2 \ldots + F_n^2 + F_{n+1}^2 = F_n \times F_{n+1} + F_{n+1}^2 = F_{n+1} \times F_{n+2}$

Proof by Induction

Roo Z

Prove That:

$$\sum_{i=1}^{n} F_i^2 = F_1^2 + F_2^2 + F_3^2 + \ldots + F_n^2 = F_n x F_{n+1}$$

Proof by Induction:

1. Check that it is true for n=1
2. Assume that it is true for n:

$$F1^2 + F2^2 + F3^2 + \ldots + Fn^2 = F_n x F_n + 1$$

3. Show that it is true for n+1

$$F1^2 + F2^2 + F3^2 + \ldots + Fn^2 + F_{n+1}^2 = F_{n+1} x F_{n+2}$$

Use the fact that $F_{n+2} = F_n + F_{n+1}$ (Definition of Fibonacci Sequence)

1. ✓ $n = 1 = 1 \times 1$

$F_1^2 + F_2^2 + \ldots + Fn^2 + F_{n+1}^2 = F_{n+1} \times F_{n+2} = F_n \times F_{n+1}$ $\quad$ $F_n \times F_{n+1} + F_{n+1}$

Work for 1 ✓ $F_1^2 + F_2^2 + \cdots F_n^2 = F_n \times F_{n+1}$

Assume for n

Prove for n+1

$$\underbrace{F_1^2 + F_2^2 + \cdots + F_n^2}_{} + F_{n+1}^2 \qquad \cancel{(F_{n-1} + F_n)^2}$$

$$= F_n \times F_{n+1} + F_{n+1}^2$$

$5 \times 2 + 2^2 =$

$2(5+2)$

$$= F_{n+1}\left(F_n + F_{n+1}\right)$$

$$= F_{n+1} F_{n+2}$$

Math Circle 11 Math League Team Round Strategy 11/13/2024 2:00 to 3:00

Cyrus, Marc, Rooz, Lizzy and Dar

Math Circle 12 Math League Team Round Strategy 12/11/2024 2:00 to 3:15

Cyrus, Marc, Rooz, Lizzy and Dar

After some discussion of why it was important for them to have a field of expertise and divide the 10 problems, each person attacked what they were best at. For this round this is really a test but by February they should be ready. I had them come up with a tentative list of the type of problems that they were best at. Using this list, I asked them to assign problems for 3 different Team Rounds and eventually they did this in less than 3 minutes. They really should not take more than 2 minutes to do this eventually Also if anyone sees a problem that is familiar to him/her, or they think they can do it easily they should immediately claim it. The goal is to find the easiest problems and do them correctly as fast as possible before moving to harder problems. So even if they get a problem assigned and they see that it too hard they should give it a grade of "0" and move on and let someone else tackle it. Cyrus and Lizzy looked at problems 6-10 amd Marc and Rooz looked at problems 1-5.

Cyrus wanted to do a round, so I let them tackle Team Round 11825. I needed to help them with many of the problems, they got the correct answer on several of them after that.

I have made the tables that they can use for the actual Round on Saturday and Cyrus will bring them. They should study the problem assignment table so they can do the first task of assignment as fast as possible.

Each person will attack the problem that they are assigned and will put the answer in the Answer box and give a grade 3 Correct, 2 May be Correct; 1 Maybe; 0 no idea. Problems that do not get a 3 should be reviewed or attacked by someone else and even the 3s should get a quick review to see that the answer is correct (I often solved the problem correct but did not give the exact answer that was Asked for.)

They also must make sure that they get all the answers into the computer in the last 5 minutes.

	#	Initial	Field of Expertise
Rooz	1	Vis	
	2	SeQ	
	3	Div	
	4	Set	
	5		

	#	Initial	Field of Expertise
Marc	1	NUM	
	2	T& D	
	3	Geo	
	4	%	
	5	Stat	

	#	Initial	Field of Expertise
Lizzy	1	Geo	
	2	Mon	
	3	Div	
	4	Seq	
	5		

	#	Initial	Field of Expertise
Cyrus	1	C&P	
	2	GEO	
	3	ALG	
	4	SEQ	
	5	%	

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#	Expertise	Assigned	Answer	G 0-3	Reviewer	Final 0-3
6	SEQ	Rooz	$\frac{2017}{2019}$	3		
7	NUM	Lizzy	891	3		
8	VIS	Rooz	35	3		
9	GEO	Cyrus				
10	STAT	Marc	9	2		

#	Expertise	Assigned	Answer	G 0-3	Reviewer	Final 0-3
1	SEQ	Rooz				
2	%	Marc				
3	C&P	Cyrus				
4	MON	Lizzy				
5	SEQ	Rooz				

	#	Initial	Field of Expertise
Cyrus	1	C&P	Combinatorics & Probability
	2	GEO	Geometry
	3	ALG	Algebra
	4	SEQ	Sequences
	5	%	Percent/Ratio//Discount/Tax/Depreciation

	#	Initial	Field of Expertise
Lizzy	1	GEO	Geometry
	2	MON	Money/Coin
	3	DIV	Divisibility/GCD/LCM
	4	SEQ	Sequences
	5		

	#	Initial	Field of Expertise
Marc	1	NUM	Number& Arithmetic&Letter Arithmetic
	2	T&D	Time/Distance/Speed
	3	GEO	Geometry
	4	%	Percent/Ratio//Discount/Tax/Depreciation
	5	STAT	Statistics (Mean and Median)

	#	Initial	Field of Expertise
Rooz	1	VIS	Visualization
	2	SEQ	Sequences
	3	DIV	Divisibility/GCD/LCM
	4	SET	Sets
	5		

Problem Assignment Table

Initial	Field of Expertise	First Choice	Second Choice	Third Choice
T&D	Time/Distance/Speed	Marc	Cyrus	Lizzy
MON	Money/Coin	Lizzy	Marc	Rooz
STAT	Statistics (Mean and Median)	Marc	Cyrus	Rooz
SET	Sets	Rooz	Marc	Lizzy
C&P	Combinatorics & Probability	Cyrus	Marc	Rooz
DIV	Divisibility/GCD/LCM	Lizzy	Rooz	Cyrus
VIS	Visualization	Rooz	Lizzy	Cyrus
NUM	Number& Arithmetic&Letter Arithmetic	Marc	Rooz	Cyrus
%	Percent/Ratio//Discount/Tax/Depreciation	Marc	Cyrus	Lizzy
GEO	Geometry	Lizzy	Cyrus	Marc
SEQ	Sequences	Rooz	Cyrus	Lizzy
ALG	Algebra	Cyrus	Marc	Lizzy

#	Expertise	Assigned	Answer	G 0-3	Reviewer	Final 0-3
1						
2						
3						
4						
5						

#	Expertise	Assigned	Answer	G 0-3	Reviewer	Final 0-3
6						
7						
8						
9						
10						

Math Circle 13, 14 Math League Exercises etc. 12/19/2024 12/26/2025 2:00 to 3:15

MC 12/19 -26/2024

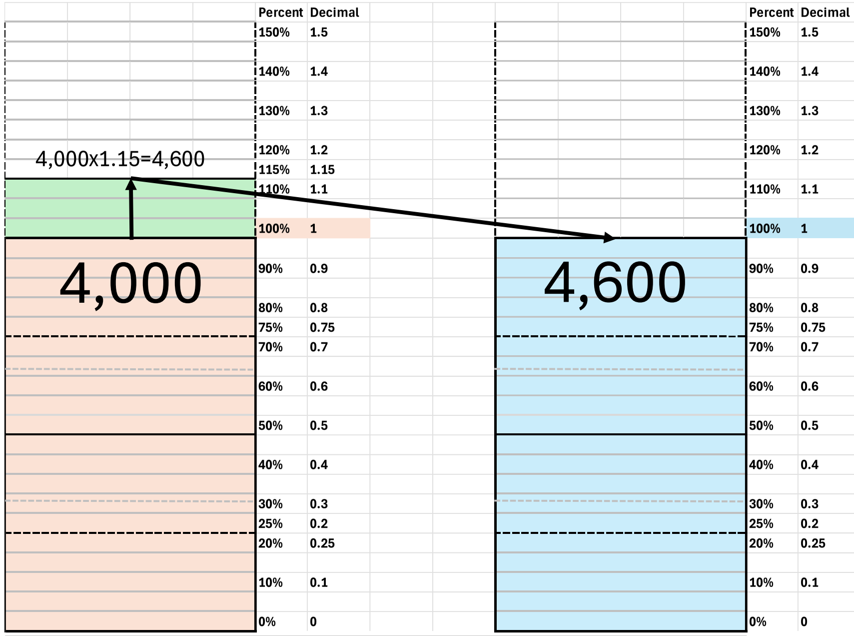
Cyrus, Rooz, and Dar

Marc was not feeling well, and we had the session with Rooz and Cyrus. We did a round of Number Sense, and I participated. Everyone did well and each of us made a few stupid mistakes!

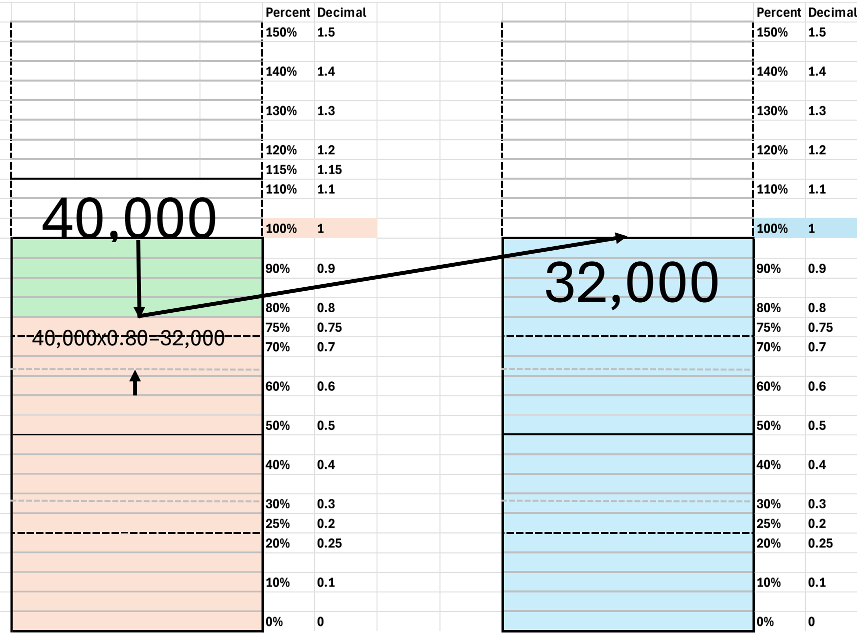
We also went over the strategies for doing percentage related problems: discounts, taxes and interest.

On 12/26 we did more exercises from Math League test.

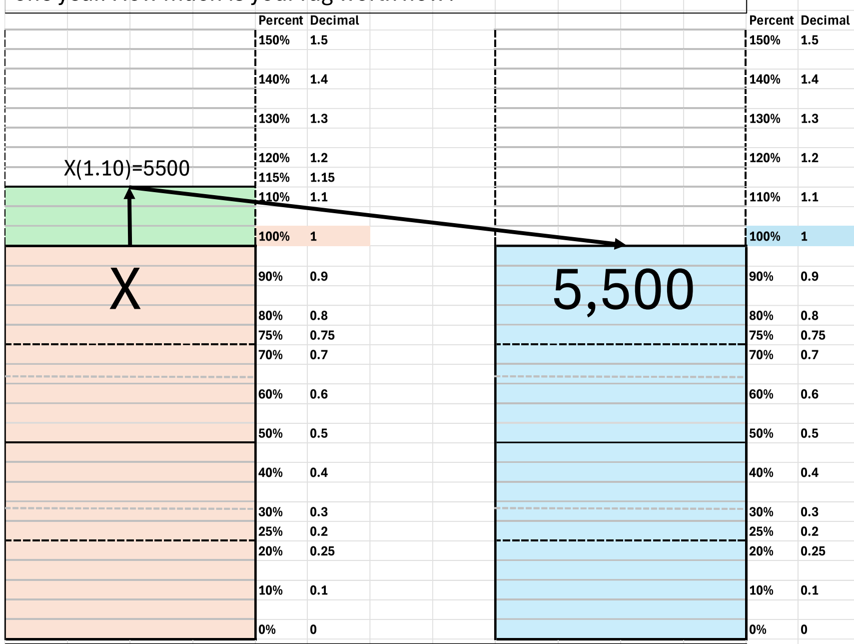
Ratio	Fraction	Percent	Decimal
3:2	3/2	150%	1.5
7:5	7/5	140%	1.4
13:10	1 3/10	130%	1.3
6:5	6/5	120%	1.2
11:10	1 1/10	110%	1.1
1:1	1/1	100%	1
9:10	9/10	90%	0.9
4:5	4/5	80%	0.8
3:4	3/4	75%	0.75
7:10	7/10	70%	0.7
3:5	3/5	60%	0.6
1:5	1/2	50%	0.5
2:5	2/5	40%	0.4
3/10	3/10	30%	0.3
1:4	1/4	25%	0.2
1:5	1/5	20%	0.25
1:10	1/10	10%	0.1
	0	0%	0



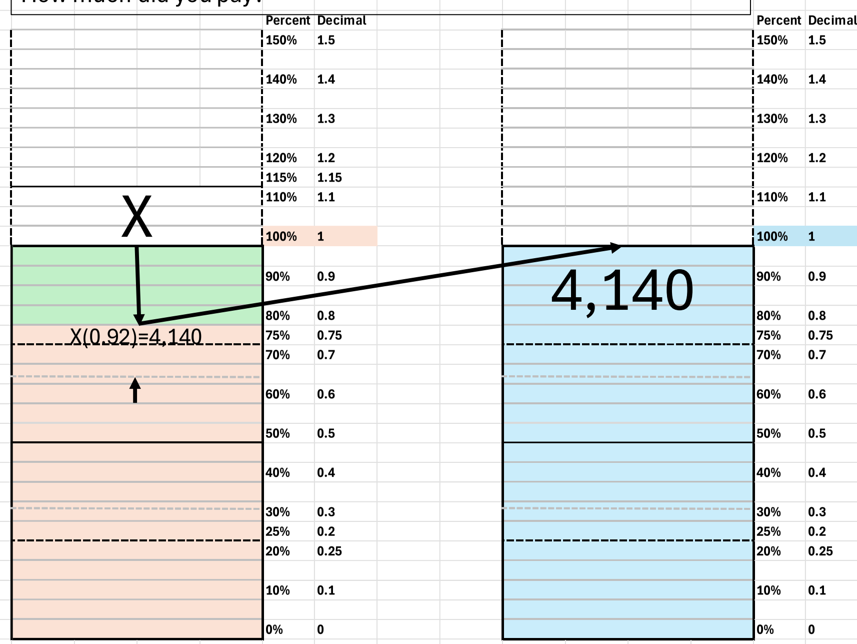
You bought a Persian rug for $4,000 and it 's value went up by 15% in one year. How much is your rug worth now?



You bought a car for $40,000 and they gave you a 20% discount. How much did you pay?



You bought stock and it's value went up by 10% in one year. How much did you pay for the stock? X(1.1)=5,500 X=5,500/1.1=5,000 X=5,000



You got an 8% discount on a refrigirator. The price after discount was $4,140. What was the original price? X(.92)=4,140 X=4140/.92=4,500

Math Circle 15 Linear Algebra 1/15/2025 2:00 to 3:15

Cyrus, Marc, Rooz, Dar

I spent the last 4 sessions working on topics from the team competition for Math League. By popular request I will leave that to Laura and Kelly and go back some othe math concepts. We started the Linear Algebra series based on 3 Blue 1 Brown series by the same name by Grant Sanderson <May he live long and prosper;=> We watched the first video and then they did the exercises on the first worksheet that I had made for them based on what Grant had covered in the video which was a great introduction to vectors and coordinate systems.

All 3 of them got started. I had to help and encourage Rooz that was a little overwhelmed at the beginning. Marc went too fast but he made a mistake with problems 3, 4, and 5. He was a bit discouraged but he corrected his error with my encouragement and finished his work with Cyrus and Rooz. Cyrus went very systematically and slow as usual and finished all the questions. Rooz finished just a little after Cyrus with a little more help from me. All 3 needed a little help with problem 5 and I had to explain it to each one of them. Exercises 1 and 2 covered drawing vectors and finding their sum. Exercises 3, 4, and 5 dealt with scaling the vectors in the first 2 problems and finding the new sums for these vectors.

We discussed also what happened to the final resulting vectors and the fact that it was turned towards the left. We also discussed that vectors A and B and A’ and B’ are perpendicular to each other.

I was hoping to cover two videos and two worksheets, but this was just right for them and took 1 ¼ hours. This has been a great review for me, and I hope to cover many of the topics in these 15 videos for them now that I have a good understanding of almost every topic;=>.

Name Cyrus Linear Algebra Problems (1)

Date 1/13/15 Draw and mark the following vectors using a black pencil:

$A\begin{bmatrix}2\\3\end{bmatrix}$ $B\begin{bmatrix}3\\-2\end{bmatrix}$ $C\begin{bmatrix}-2\\2\end{bmatrix}$ $D\begin{bmatrix}-1\\-4\end{bmatrix}$

2. Find, draw, and mark the following vectors with red pencil.

$\vec{U} = \vec{A} + \vec{B}$

$\vec{V} = \vec{C} + \vec{D}$

$\vec{P} = \vec{U} + \vec{V}$

3. Find, draw and mark the following vectors with black pencil

$\vec{A}' = 2\vec{A}$ $\vec{B}' = 2\vec{B}$ $\vec{C}' = 3C$ $\vec{D}' = 3D$

4. Find, draw and mark the following vector with red pencil.

$\vec{U}' = \vec{A}' + \vec{B}'$

$\vec{V}' = \vec{C}' + \vec{D}'$

$\vec{P}' = \vec{U}' + \vec{V}'$

5. Find a and b so that $\vec{P}' = a\vec{U} + b\vec{V}$

$a=2$ $b=3$

Cyrus 1/15/25

hand2mind

1cm METRIC GRAPH PAPER

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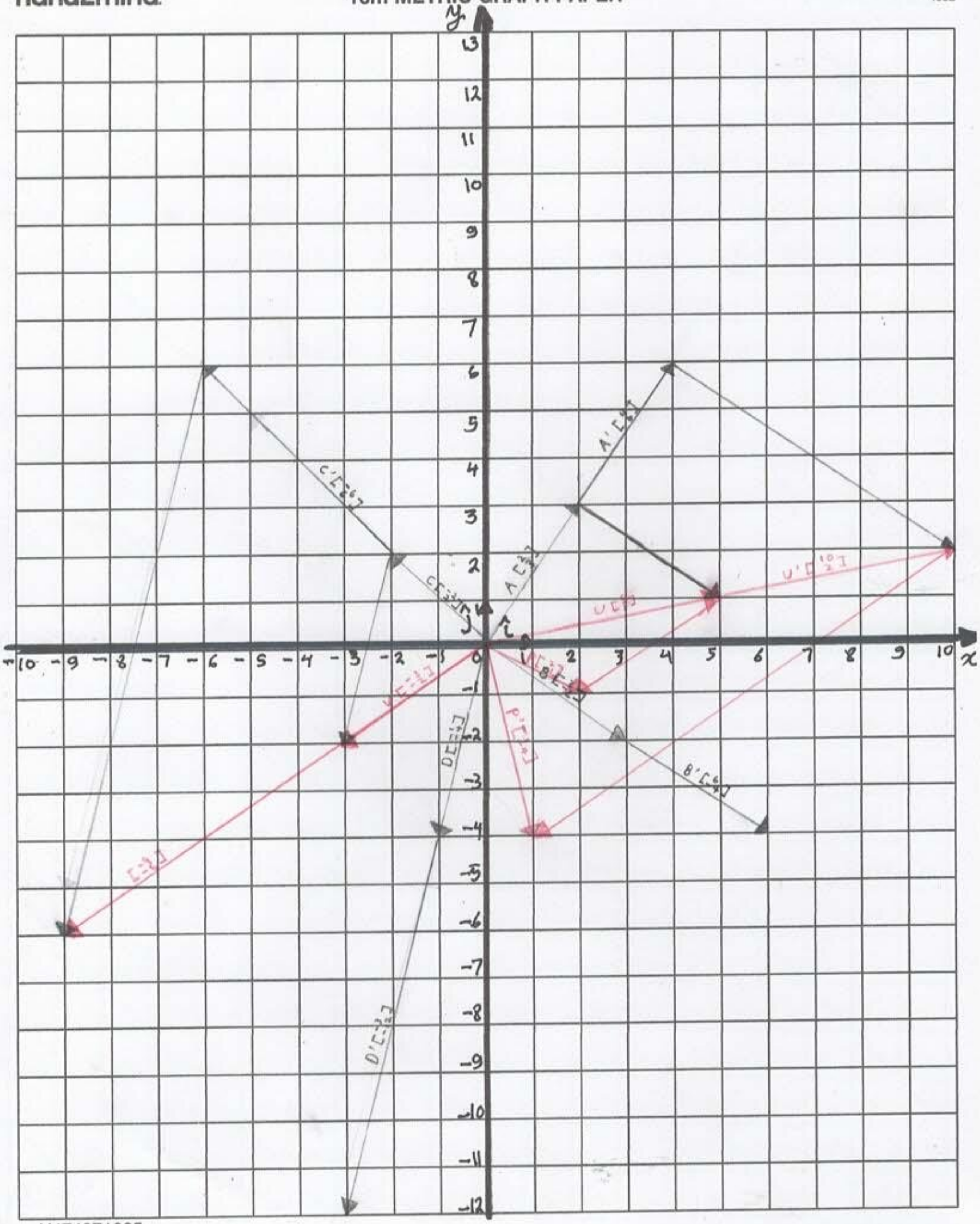

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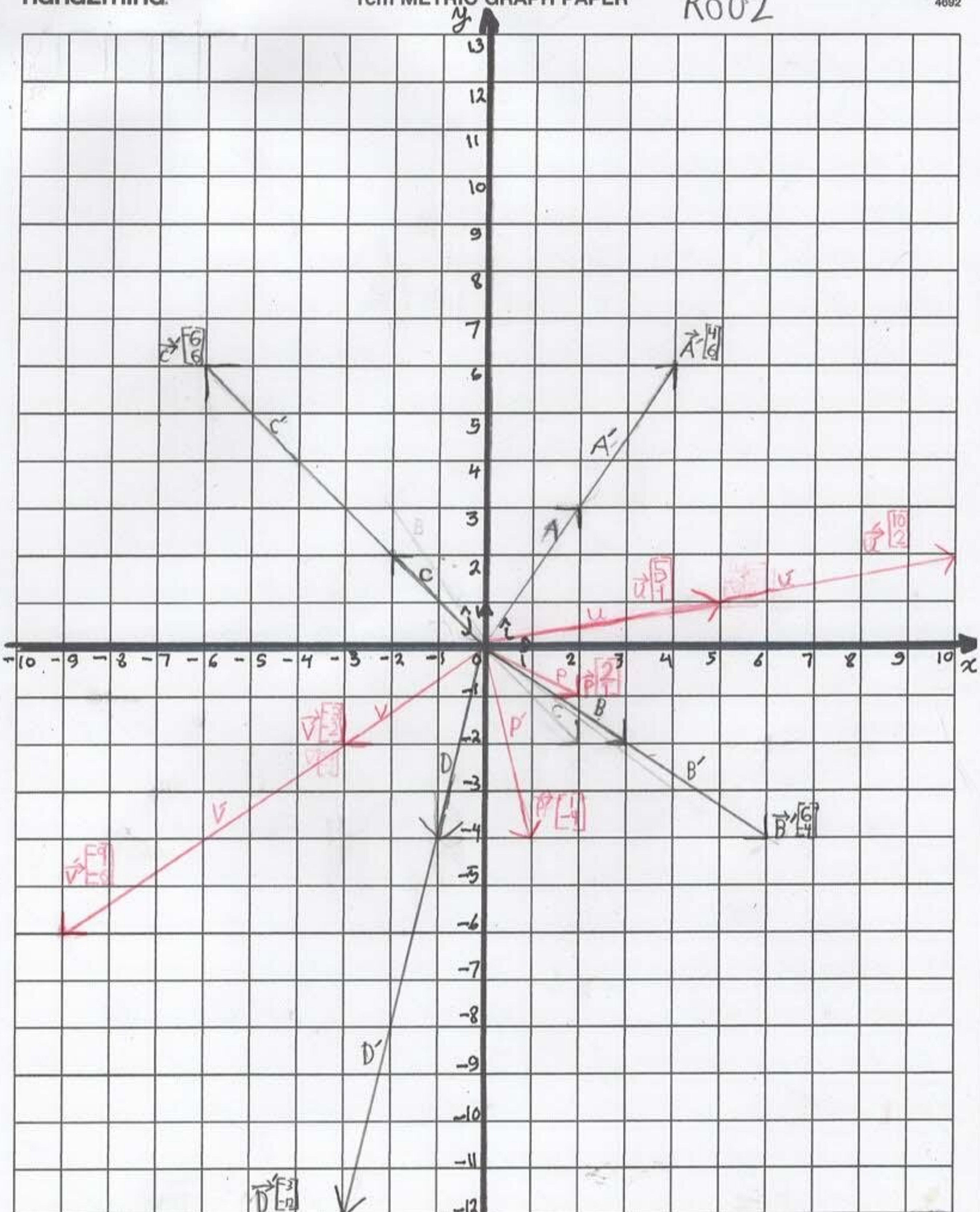

y
x
A
A'
B
B'
C
C'
D
D'
u
v
p
p'

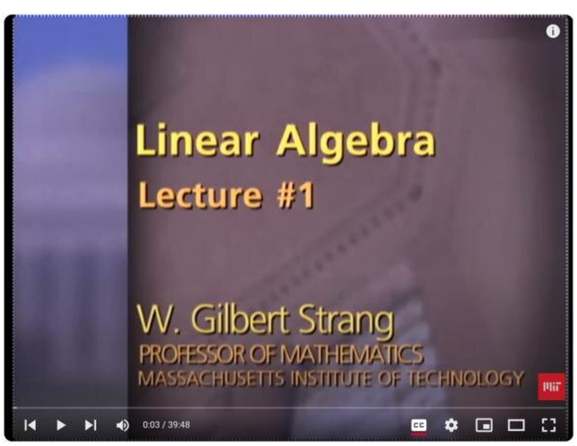
Math Circle 16 Linear Algebra 2 1/29/2025 2:00 to 3:15

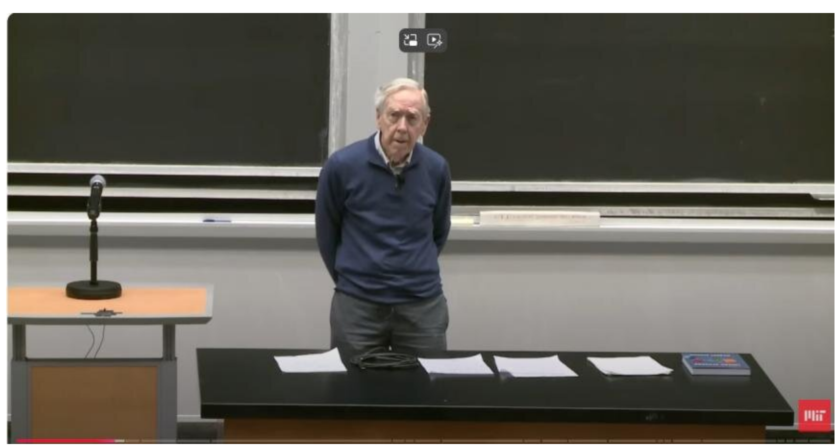
Cyrus, Marc, Rooz, Dar

We continued the [Linear Algebra] series based on 3 Blue 1 Brown series by the same name by Grant Sanderson. We watched the second video, and I also did a few demonstrations for them also. We also covered a bit of the 3ed video that we will cover fully next week. Then they did the exercises on the second worksheet that I had made for them based on what Grant had covered in the 2ed video. This is a great introduction to vectors, matrices and vector spaces. I have started to work on MIT Course on Linear Algebra by Gilbert Strand that is supposed to be the father of Linear Algebra. It is really great for me, and I will extract a lot of the main topics for them. I have also ordered his book.

Everyone did well. Marc went slower and his sheet is great. I encourage and pull Rooz along but he did quite well. Cyrus' work was great as usual. They asked for a challenge and did a a rotation of Z (2,1) something that we go into details next session.

It was one of our best sessions and they did well.





His Final Lecture 2023

Name

Date

Linear Algebra Problems (2)

1. Find, draw and mark the following new Basis vectors:

$$\vec{U}\begin{bmatrix}4\\4\end{bmatrix} \quad \vec{V}\begin{bmatrix}-2\\3\end{bmatrix}$$

2. Write $\vec{U}$ and $\vec{V}$ in terms of the Basis vectors $\hat{\imath}$ and $\hat{\jmath}$

3. Find, draw and mark the sum of $\vec{U}$ and $\vec{V}$: $\vec{X} = \vec{U} + \vec{V}$

4. ~~F~~Write $\vec{X}$ as a linear combination of $\hat{\imath}$ and $\hat{\jmath}$ and verify that the scalars that you find are The coordinates of $\vec{X}$.

5. Find, draw, and Mark the vector $\vec{Y} = -\vec{X}$ and write it in terms of basis vectors $\hat{\imath}$ and $\hat{\jmath}$

Cyrus 1/29/25

$\vec{U} = 4\hat{\imath} + 4\hat{\jmath}$ $\vec{Y} = -2\hat{\imath} + -7\hat{\jmath}$

$\vec{V} = -2\hat{\imath} + 3\hat{\jmath}$

$\vec{X} = 2\hat{\imath} + 7\hat{\jmath}$

hand2mind

1cm METRIC GRAPH PAPER

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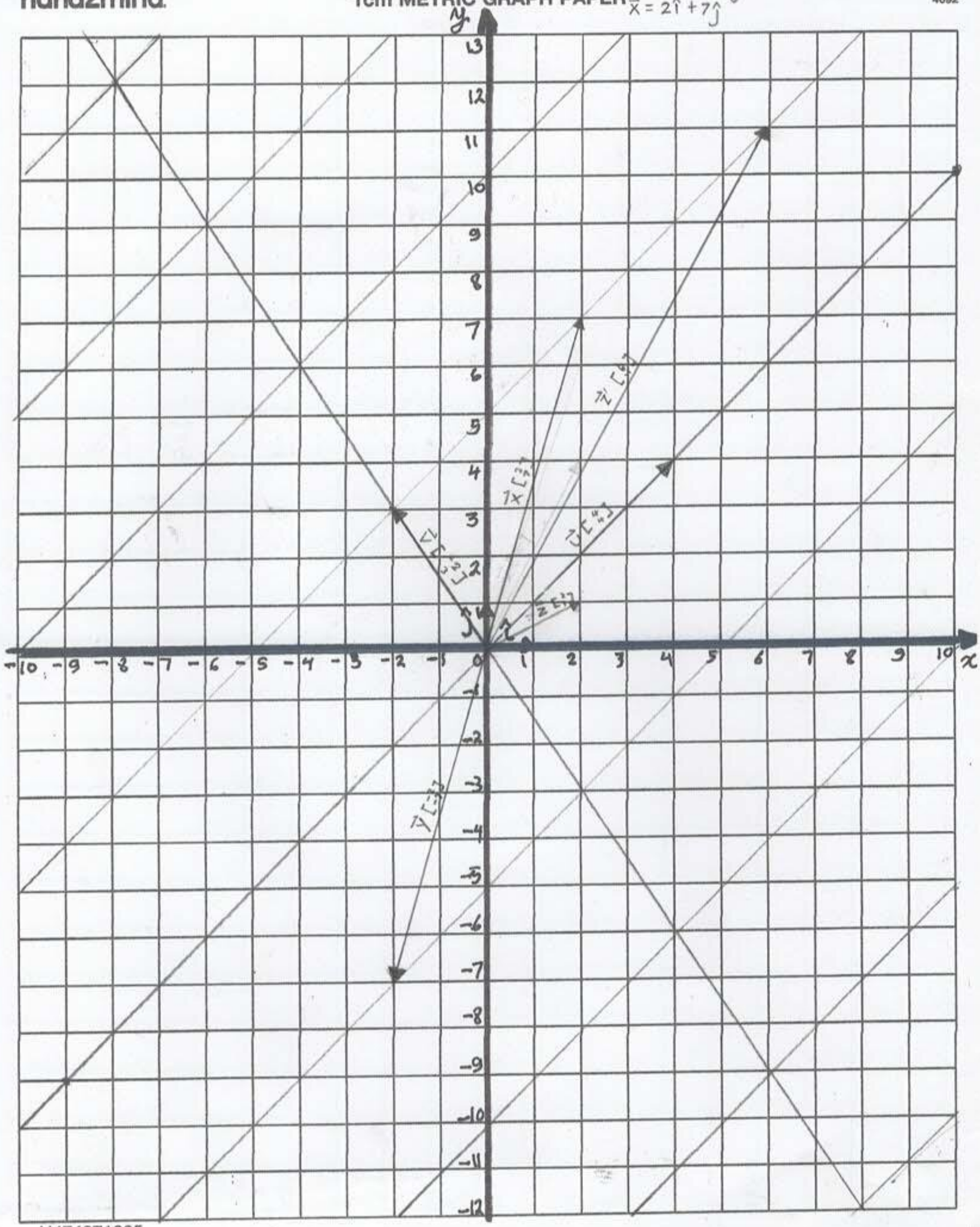

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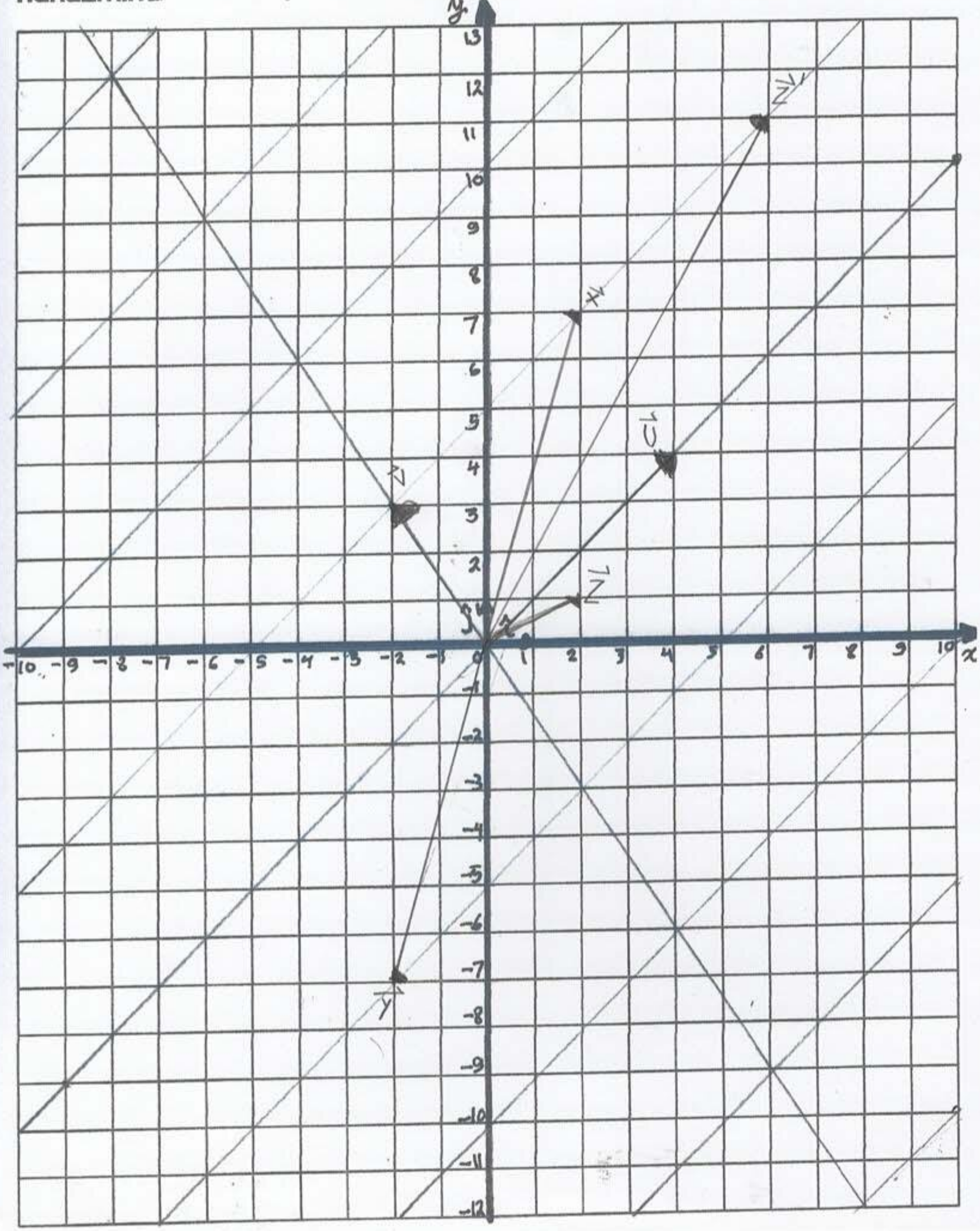
hand2mind. Marc **1cm METRIC GRAPH PAPER** 4692



Rooz

hand2mind

1cm METRIC GRAPH PAPER

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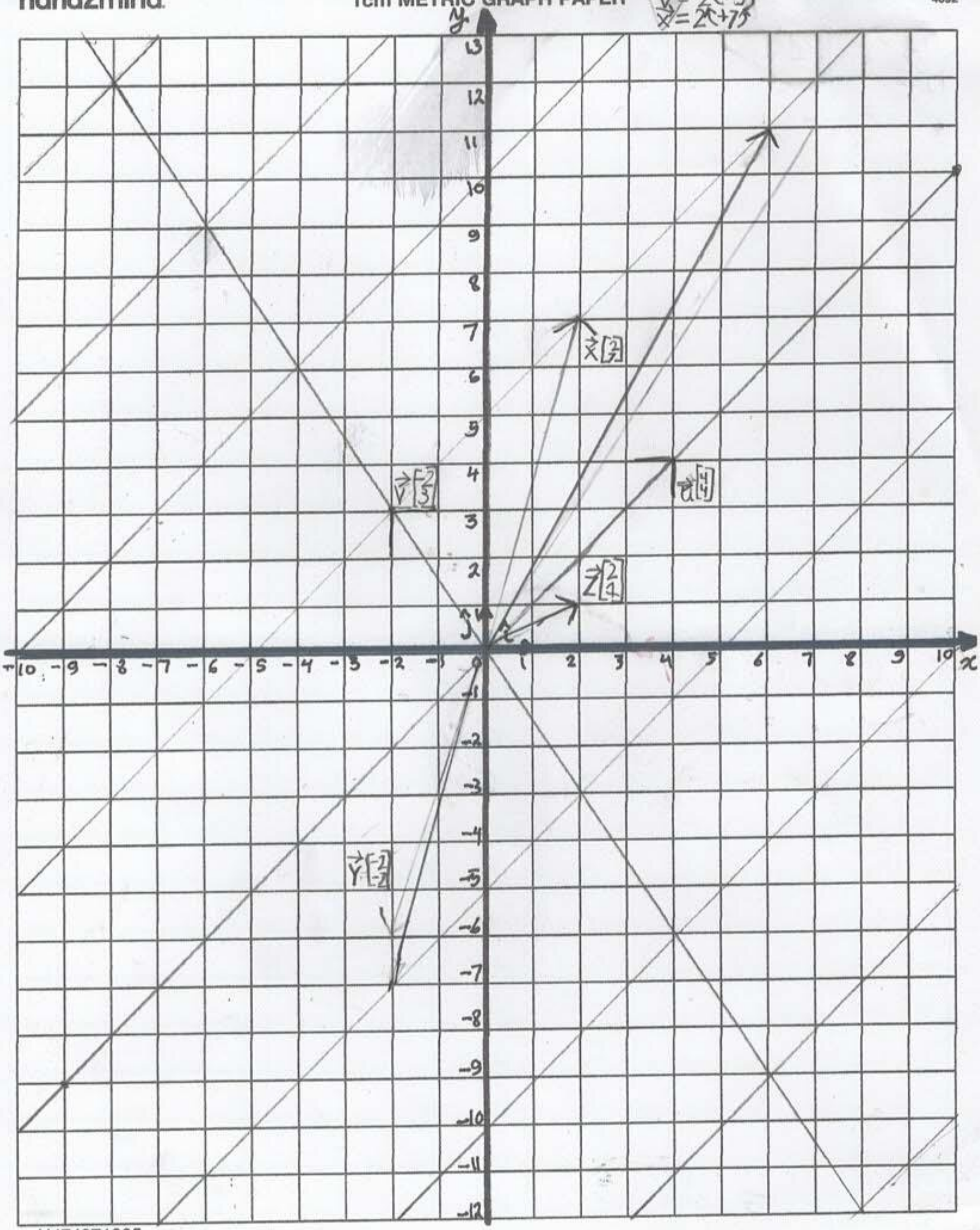
$\vec{u}=4\hat{i}+4\hat{j}$
$\vec{v}=-2\hat{i}+3\hat{j}$
$\vec{x}=2\hat{i}+7\hat{j}$



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Math Circle 17 Linear Algebra 3 2/04/2025 2:00 to 3:00

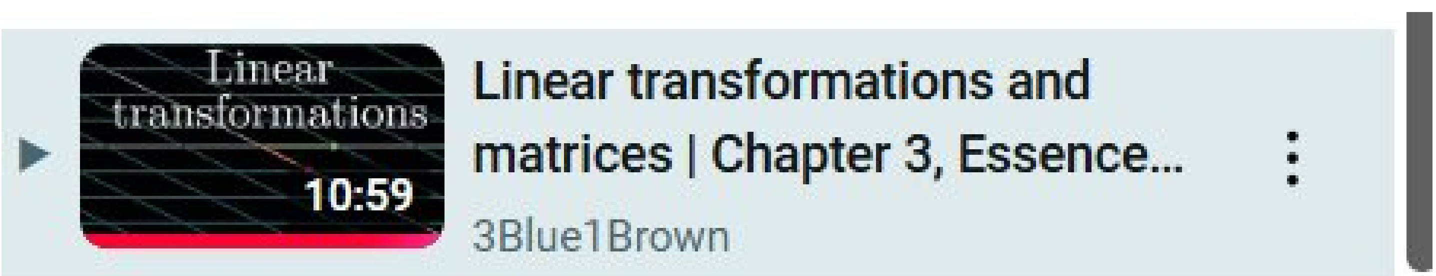
Cyrus, Marc, Rooz, Dar

We continued the Linear Algebra series based on 3 Blue 1 Brown series by the same name by Grant Sanderson. We watched the third, and as before I also did a few demonstrations for them as I felt it was needed. Then they did the exercises on the third worksheet that I had prepared for them based on what Grant had covered in the 3ed video. This is a great way to understand matrices as translations of vectors. I got the book from Strang MIT lectures and have done 11 sessions and will do them all of them eventually to hunt for good stuff for the kids.

Everyone did well. Marc went slower and but made a mistake but did not get too upset and completed every task greatly. I encouraged and pulled Rooz along, but he did quite well and I believe he gets it. Cyrus' work was methodical and great as usual.

At the end we used Cyrus' computer and mine for them to work on the two Python programs that I had made for them to translate the vectors and shapes. Cyrus and Marc worked together and I worked with Rooz. They liked the programs but they were not as impressed as I thought they would be! I guess I was so excited that I could make programs like that using ChatGPT that I thought they would like it as much;< I am attaching the code that Mihai can put on a computer and have Marc play with it.

It was one of the very good sessions, nevertheless and I think they really see the relation between the Matrix and the transformation that it represents.



Name: Marc
Date:

Linear Algebra Problems (3)

1. Find, draw and mark the following new basis vectors in red:
$$\hat{i}' = \begin{bmatrix}2\\1\end{bmatrix} \qquad \hat{j}' = \begin{bmatrix}-2\\1\end{bmatrix}$$

2. Find the trasformation matrix $M = [\hat{i}'\ \hat{j}']$

$$\begin{bmatrix}a & c\\ b & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}ax+cy\\bx+dy\end{bmatrix}$$

$$M = \begin{bmatrix}2 & -2\\1 & 1\end{bmatrix}$$

3. Find, draw and mark the following vectorsin in black:
$$\vec{U} = \begin{bmatrix}3\\4\end{bmatrix} \text{ and } \vec{V} = \begin{bmatrix}-2\\-5\end{bmatrix}$$

4. Find, draw and mark the transformet Vectors in red:
$$\vec{U}' = 3\hat{i}' + 4\hat{j}' = M\begin{bmatrix}3\\4\end{bmatrix}$$
$$\begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}-8\\4\end{bmatrix}\begin{bmatrix}-2\\7\end{bmatrix}$$

$$\vec{V}' = -2\hat{i}' - 5\hat{j}' = M\begin{bmatrix}-2\\-5\end{bmatrix}$$
$$\begin{bmatrix}-4\\-2\end{bmatrix} + \begin{bmatrix}10\\-5\end{bmatrix} = \begin{bmatrix}6\\-7\end{bmatrix}$$

5. On the transformed coordinate system verify that $\vec{U}' = 3\hat{i}' + 4\hat{j}'$ and
yes
$$\vec{V}' = -2\hat{i}' - 5\hat{j}'$$
yes

Hint find and Mark
$3\hat{i}'$ and $4\hat{j}'$
$-2\hat{i}'$ ad $-5\hat{j}'$

hand2mind Marc

1cm METRIC GRAPH PAPER

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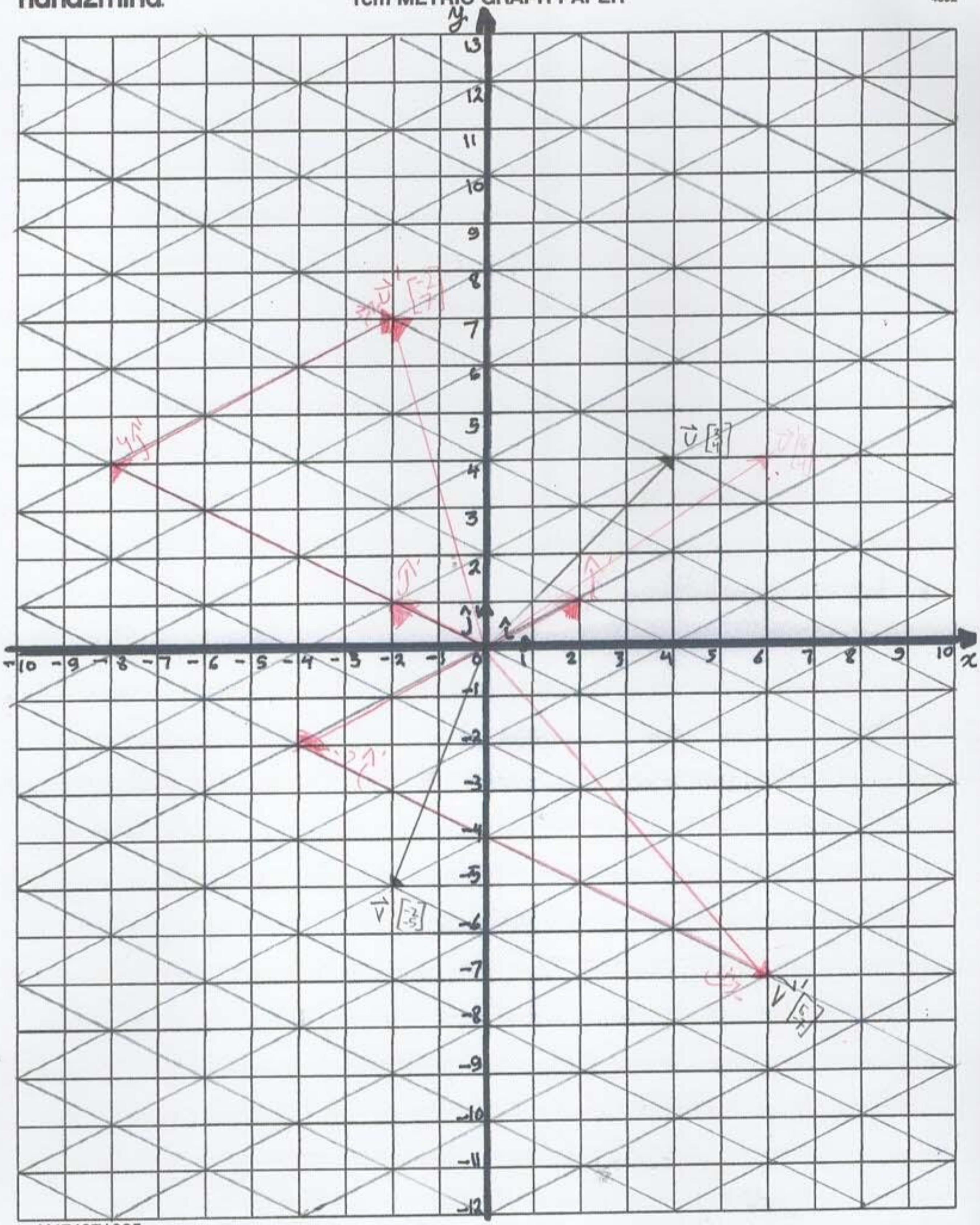

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Name: ROOZ
Date:

Linear Algebra Problems (3)

1. Find, draw and mark the following new basis vectors in red:

$$\hat{i}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \hat{j}' = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. Find the trasformation matrix $M = [\hat{i}' \ \hat{j}']$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+cy \\ bx+dy \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

3. Find, draw and mark the following vectorsin in black:

$$\vec{U} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } \vec{V} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

4. Find, draw and mark the transformet vectors in red:

$$\vec{U}' = 3\hat{i}' + 4\hat{j}' = M \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\vec{U} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -8 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

$$\vec{V}' = -2\hat{i}' - 5\hat{j}' = M \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

$$\vec{V} = -2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 5\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

5. On the transformed coordinate system verify that $\vec{U}' = 3\hat{i}' + 4\hat{j}'$ and

$$\vec{V}' = -2\hat{i}' - 5\hat{j}'$$

Hint find and Mark
$3\hat{i}'$ and $4\hat{j}'$
$-2\hat{i}'$ ad $-5\hat{j}'$

ROOZ

hand2mind

1cm METRIC GRAPH PAPER

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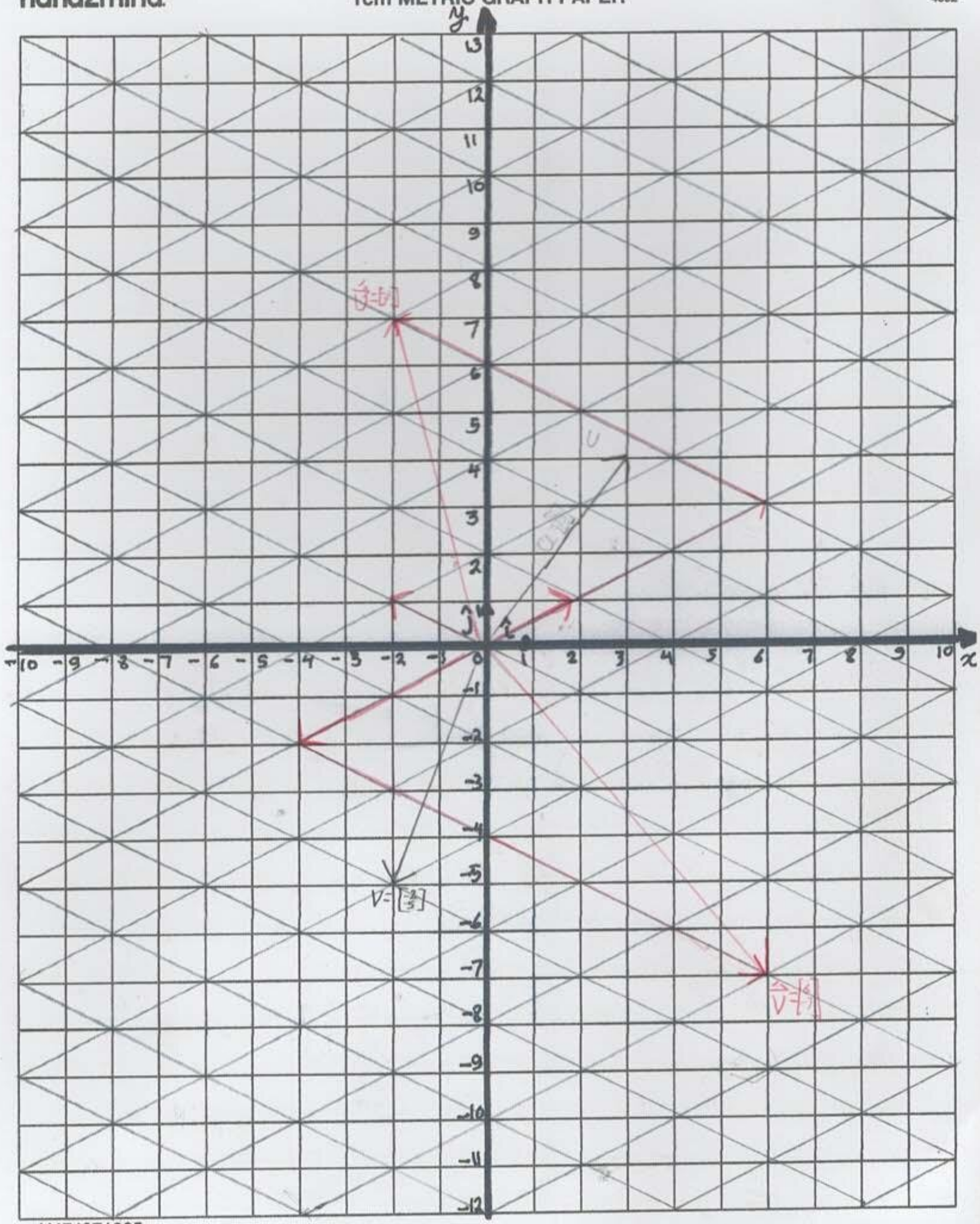

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Name: Cyrus
Date: 2/5/25

Linear Algebra Problems (3)

1. Find, draw and mark the following new basis vectors in red:

$$\hat{i}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \hat{j}' = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. Find the trasformation matrix $M = [\hat{i}' \; \hat{j}']$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+cy \\ bx+dy \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

3. Find, draw and mark the following vectorsin in black:

$$\vec{U} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } \vec{V} = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

4. Find, draw and mark the transformet vectors in red:

$$\vec{U}' = 3\hat{i}' + 4\hat{j}' = M \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\vec{V}' = -2\hat{i}' - 5\hat{j}' = M \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

5. on the transformed coordinate system verify that $\vec{U}' = 3\hat{i}' + 4\hat{j}'$ and

$$\vec{V}' = -2\hat{i}' - 5\hat{j}'$$

Hint find and Mark
$3\hat{i}'$ and $4\hat{j}'$
$-2\hat{i}'$ ad $-5\hat{j}'$

$$\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6+(-8) \\ 3+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

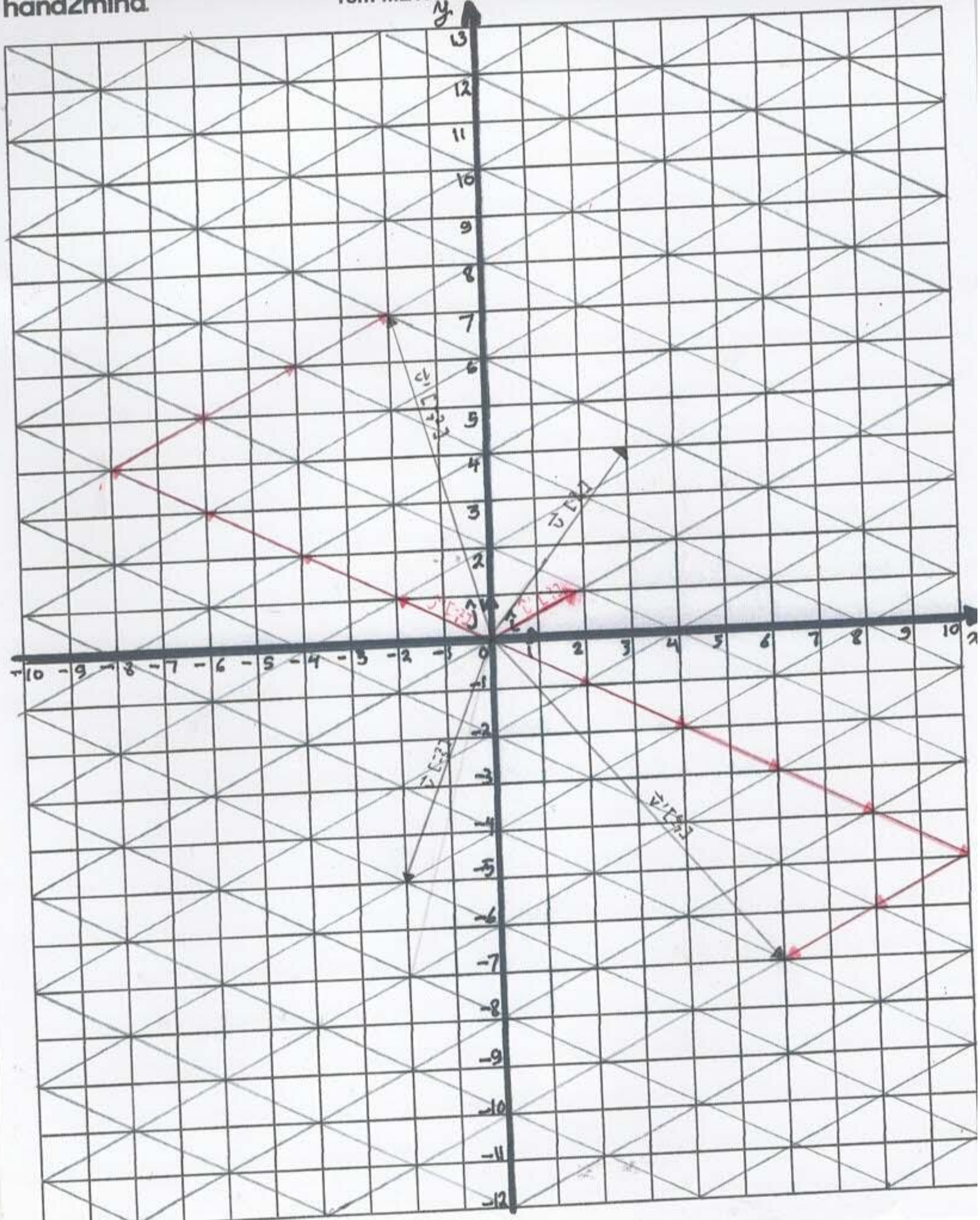
$$\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 4+10 \\ -2+(-5) \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

Cyrus 2/5/25

4692

hand2mind

1cm METRIC GRAPH PAPER



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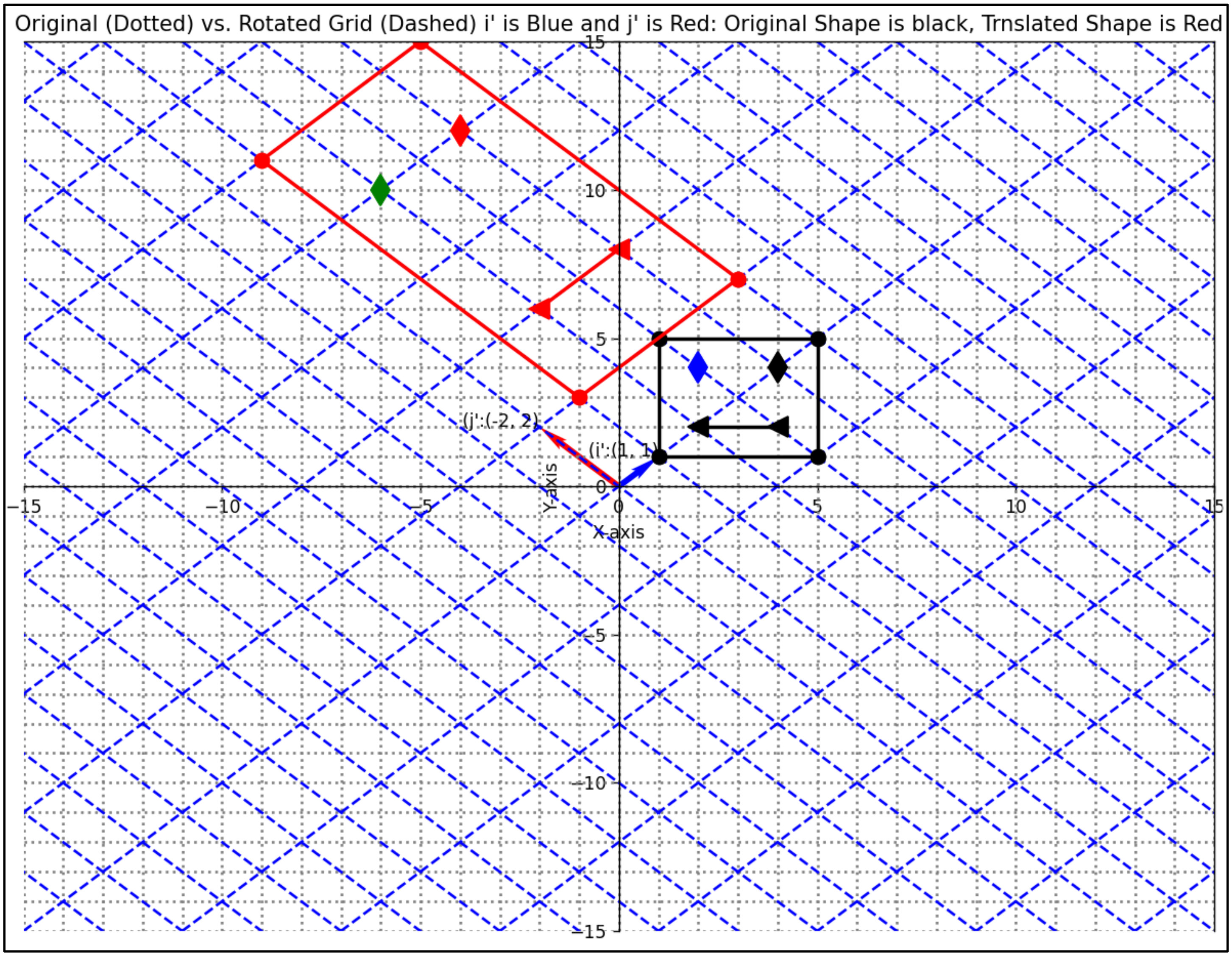
Shape and Axis translation



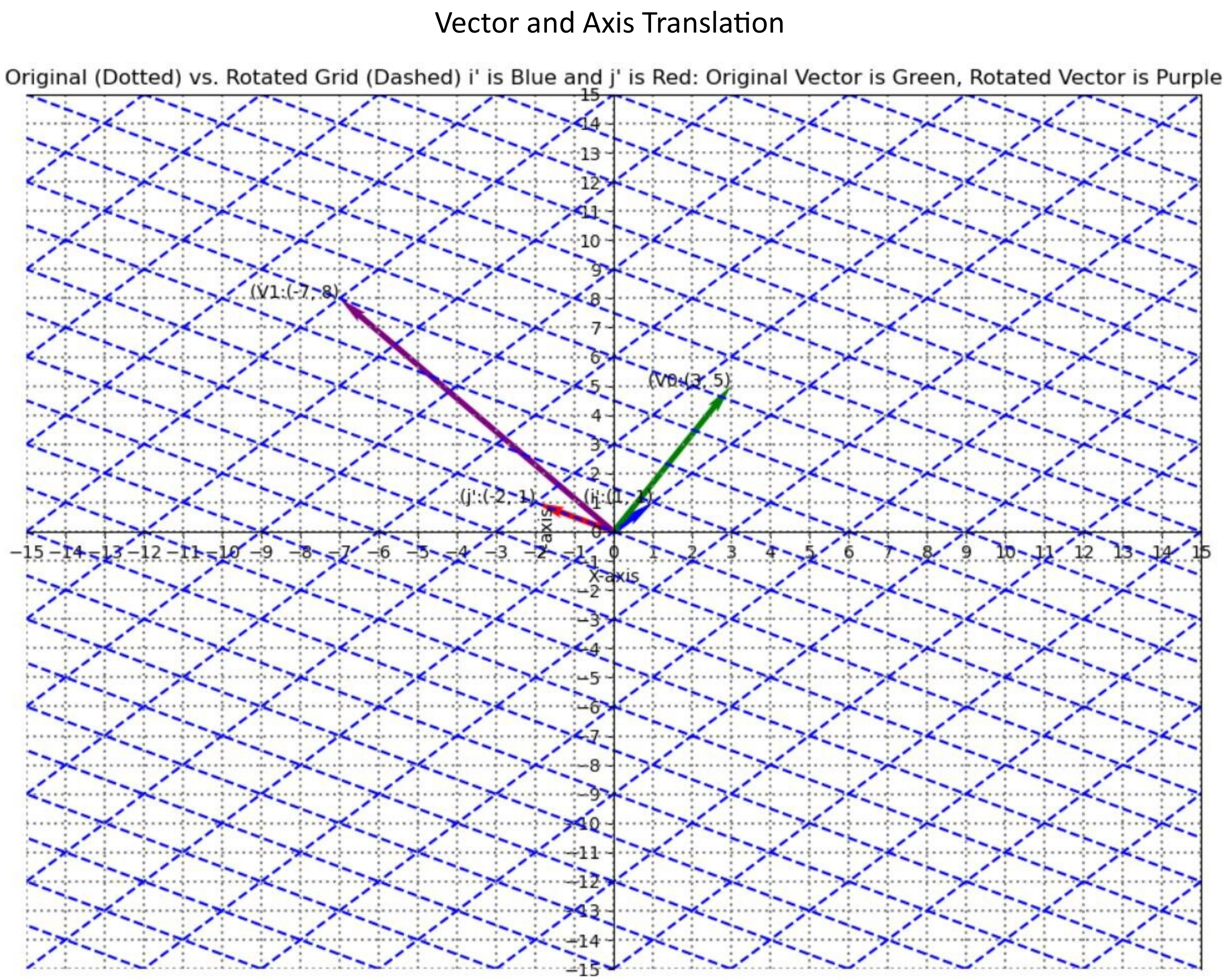

```
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
import matplotlib.ticker as ticker
matplotlib.use("TkAgg")  # Force using Tkinter for separate window

# ****************************************************************************
# ****************************************************************************
# **************define your i' and j^ and the points you want to transform **************

M = np.array([[0,1], [-1,0]] ) # Row Vectors for Tranlation needs to be Transposed
Mij = np.transpose(M)        # column vectors for the Translation

A = np.array([1,1]) # First corner
B = np.array([1,5]) # Second corner
C = np.array([5,5]) # Third corner
D = np.array([5,1]) # Fourth corner
```

```
E = np.array([2,2]) # Mouth Left
F = np.array([4,2]) # Mouth Right
G = np.array([2,4]) # Eye Left
H = np.array([4,4]) # Eye Right

# **************************************************************************************
# **************************************************************************************

# Function to create grid points
def create_grid(x_range, y_range, step=1):
    x = np.arange(x_range[0], x_range[1] + step, step)
    y = np.arange(y_range[0], y_range[1] + step, step)
    grid_x, grid_y = np.meshgrid(x, y)
    return grid_x, grid_y

# Function to apply a rotation matrix
def rotate_points(x, y):
    R = np.array(Mij)  # Rotation matrix
    xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
    return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# def rotate_points(x, y, angle_degrees):
#    theta = np.radians(angle_degrees)  # Convert to radians
#    R = np.array([[np.cos(theta), -np.sin(theta)], [np.sin(theta), np.cos(theta)]])  # Rotation matrix
#    xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
#    return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# Create a grid
grid_x, grid_y = create_grid((-15, 15), (-15, 15), step=1)

# Rotate the grid by Matrix M degrees
# rotated_x, rotated_y = rotate_points(grid_x, grid_y, 30)
rotated_x, rotated_y = rotate_points(grid_x, grid_y)

# Plot original and rotated grid
#fig, ax  # Check if figure exists

fig, ax = plt.subplots(figsize=(12, 12))

# Draw original grid
for i in range(grid_x.shape[0]):
    ax.plot(grid_x[i, :], grid_y[i, :], 'gray', linestyle='dotted')  # Horizontal lines
for j in range(grid_x.shape[1]):
    ax.plot(grid_x[:, j], grid_y[:, j], 'gray', linestyle='dotted')  # Vertical lines
```

```
# Draw rotated grid
for i in range(rotated_x.shape[0]):
    ax.plot(rotated_x[i, :], rotated_y[i, :], 'b', linestyle='dashed')  # Horizontal lines
for j in range(rotated_x.shape[1]):
    ax.plot(rotated_x[:, j], rotated_y[:, j], 'b', linestyle='dashed')  # Vertical lines

ax.set_xlim(-15, 15)
ax.set_ylim(-15, 15)
ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_title("Original (Dotted) vs. Rotated Grid (Dashed) i' is Blue and j' is Red: Original Shape is black, Trnslated Shape is Red")
ax.grid(False)  # Hide default Matplotlib grid
# Your result vector after rotation
# Mij i' and j' as a column Matrix
Mij = np.transpose(M) # column vectors

A1 = np.matmul(Mij, A)  # Matrix multiplication
B1 = np.matmul(Mij, B)  # Matrix multiplication
C1 = np.matmul(Mij, C)  # Matrix multiplication
D1 = np.matmul(Mij, D)  # Matrix multiplication
E1 = np.matmul(Mij, E)  # Matrix multiplication
F1 = np.matmul(Mij, F)  # Matrix multiplication
G1 = np.matmul(Mij, G)  # Matrix multiplication
H1 = np.matmul(Mij, H)  # Matrix multiplication

ox, oy = [0], [0]
x, y = M[0,0], M[0,1]
z, w = M[1,0], M[1,1]

#print("   i' j'")
#print(Mij)
print("A",A)
print("B",B)
print("C",C)
print("D",D)
print("E",E)
print("F",F)
print("G",G)
print("H",H)
print("A1",A1)
print("B1",B1)
print("C1",C1)
print("D1",D1)
print("E1",E1)
```

```
print("F1",F1)
print("G1",G1)
print("H1",H1)

#fig, ax = plt.subplots(figsize=(12, 12))
plt.quiver(ox, oy, x, y, angles='xy', scale_units='xy', scale=1, color='blue', width=0.005, headwidth=2)
plt.quiver(ox, oy, z, w, angles='xy', scale_units='xy', scale=1, color='red', width=0.005, headwidth=2)

plt.plot([A[0],B[0]], [A[1],B[1]], color="black", linewidth=2, marker="o", markersize=8, label="Line Segment")
plt.plot([B[0],C[0]], [B[1],C[1]], color="black", linewidth=2, marker="o", markersize=8, label="Line Segment")
plt.plot([C[0],D[0]], [C[1],D[1]], color="black", linewidth=2, marker="o", markersize=8, label="Line Segment")
plt.plot([D[0],A[0]], [D[1],A[1]], color="black", linewidth=2, marker="o", markersize=8, label="Line Segment")
plt.plot([E[0],F[0]], [E[1],F[1]], color="black", linewidth=2, marker="<", markersize=12, label="Line Segment")
plt.plot([G[0],G[0]], [G[1],G[1]], color="blue", linewidth=2, marker="d", markersize=12, label="Line Segment")
plt.plot([H[0],H[0]], [H[1],H[1]], color="black", linewidth=2, marker="d", markersize=12, label="Line Segment")
plt.plot([A1[0],B1[0]], [A1[1],B1[1]], color="red", linewidth=2, marker="o", markersize=8, label="Line
Segment")
plt.plot([B1[0],C1[0]], [B1[1],C1[1]], color="red", linewidth=2, marker="o", markersize=8, label="Line
Segment")
plt.plot([C1[0],D1[0]], [C1[1],D1[1]], color="red", linewidth=2, marker="o", markersize=8, label="Line
Segment")
plt.plot([D1[0],A1[0]], [D1[1],A1[1]], color="red", linewidth=2, marker="o", markersize=8, label="Line
Segment")
plt.plot([E1[0],F1[0]], [E1[1],F1[1]], color="red", linewidth=2, marker="<", markersize=12, label="Line
Segment")
plt.plot([G1[0],G1[0]], [G1[1],G1[1]], color="green", linewidth=2, marker="d", markersize=12, label="Line
Segment")
plt.plot([H1[0],H1[0]], [H1[1],H1[1]], color="red", linewidth=2, marker="d", markersize=12, label="Line
Segment")
#plt.draw()  # Update the plot
#plt.pause(0.5)  # Small pause to allow updates

# Labels and title
#plt.xlabel("X-axis")
#plt.ylabel("Y-axis")

ax.spines['left'].set_position('zero')  # Y-axis at x=0
ax.spines['bottom'].set_position('zero')  # X-axis at y=0
plt.xlim(-15, 15)  # Set x-axis limits from 0 to 10
plt.ylim(-15, 15)  # Set y-axis limits from 0 to 60
#ax.xaxis.set_major_locator(ticker.MultipleLocator(1))  # Ticks every 1 units
#ax.yaxis.set_major_locator(ticker.MultipleLocator(1))  # Ticks every 1 units


# indicate the value for all the vectors
```

```
plt.text(x, y, f"({"i':("}{x}, {y})", fontsize=10, ha='right')
plt.text(z, w, f"({"j':("}{z}, {w})", fontsize=10, ha='right')

# plt.Show the plot in a new interactive window
```




```
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
import matplotlib.ticker as ticker
matplotlib.use("TkAgg")  # Force using Tkinter for separate window

#
**********************************************************************************************
*********
# define your two new i6 and j^ and the vector you want to rotate **********************************
M   = np.array([[3,2], [-4,-1]] ) # row vectors
```

```
V0 = np.array([3,5])
#
*********************************************************************************************
********
#
*********************************************************************************************
*********

Mij = np.transpose(M) # column vectors

# Function to create grid points
def create_grid(x_range, y_range, step=1):
    x = np.arange(x_range[0], x_range[1] + step, step)
    y = np.arange(y_range[0], y_range[1] + step, step)
    grid_x, grid_y = np.meshgrid(x, y)
    return grid_x, grid_y

# Function to apply a rotation matrix
def rotate_points(x, y):
    R = np.array(Mij)  # Rotation matrix
    xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
    return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# def rotate_points(x, y, angle_degrees):
#    theta = np.radians(angle_degrees)  # Convert to radians
#    R = np.array([[np.cos(theta), -np.sin(theta)], [np.sin(theta), np.cos(theta)]])  # Rotation matrix
#    xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
#    return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# Create a grid
grid_x, grid_y = create_grid((-15, 15), (-15, 15), step=1)

# Rotate the grid by Matrix M degrees
# rotated_x, rotated_y = rotate_points(grid_x, grid_y, 30)
rotated_x, rotated_y = rotate_points(grid_x, grid_y)

# Plot original and rotated grid
#fig, ax  # Check if figure exists

fig, ax = plt.subplots(figsize=(12, 12))

# Draw original grid
for i in range(grid_x.shape[0]):
    ax.plot(grid_x[i, :], grid_y[i, :], 'gray', linestyle='dotted')  # Horizontal lines
for j in range(grid_x.shape[1]):
```

```
    ax.plot(grid_x[:, j], grid_y[:, j], 'gray', linestyle='dotted')  # Vertical lines

# Draw rotated grid
for i in range(rotated_x.shape[0]):
    ax.plot(rotated_x[i, :], rotated_y[i, :], 'b', linestyle='dashed')  # Horizontal lines
for j in range(rotated_x.shape[1]):
    ax.plot(rotated_x[:, j], rotated_y[:, j], 'b', linestyle='dashed')  # Vertical lines

ax.set_xlim(-15, 15)
ax.set_ylim(-15, 15)
ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_title("Original (Dotted) vs. Rotated Grid (Dashed) i' is Blue and j' is Red: Original Vector is Green, Rotated
Vector is Purple")
ax.grid(False)  # Hide default Matplotlib grid
# Your result vector after rotation
# Mij i' and j' as a column Matrix
Mij = np.transpose(M) # column vectors
V1 = np.matmul(Mij, V0)  # Matrix multiplication

ox, oy = [0], [0]
x, y = M[0,0], M[0,1]
z, w = M[1,0], M[1,1]
v0x, v0y = V0[0], V0[1]
v1x, v1y= V1[0], V1[1]

#print("   i' j'")
#print(Mij)
#print("V0",V0)
#print("V1",V1)



#fig, ax = plt.subplots(figsize=(12, 12))
plt.quiver(ox, oy, x, y, angles='xy', scale_units='xy', scale=1, color='blue', width=0.005, headwidth=2)
#plt.draw()  # Update the plot
#plt.pause(0.5)  # Small pause to allow updates
plt.quiver(ox, oy, z, w, angles='xy', scale_units='xy', scale=1, color='red', width=0.005, headwidth=2)
#plt.draw()  # Update the plot
#plt.pause(0.5)  # Small pause to allow updates
plt.quiver(ox, oy, v0x, v0y, angles='xy', scale_units='xy', scale=1, color='green', width=0.005, headwidth=2)
plt.draw()  # Update the plot
plt.pause(0.5)  # Small pause to allow updatesplt.
plt.quiver(ox, oy, v1x, v1y, angles='xy', scale_units='xy', scale=1, color='purple', width=0.005, headwidth=2)
#plt.draw()  # Update the plot
#plt.pause(0.5)  # Small pause to allow updates
```

```
# Labels and title
#plt.xlabel("X-axis")
#plt.ylabel("Y-axis")

ax.spines['left'].set_position('zero')  # Y-axis at x=0
ax.spines['bottom'].set_position('zero')  # X-axis at y=0
plt.xlim(-15, 15)  # Set x-axis limits from 0 to 10
plt.ylim(-15, 15)  # Set y-axis limits from 0 to 60
ax.xaxis.set_major_locator(ticker.MultipleLocator(1))  # Ticks every 1 units
ax.yaxis.set_major_locator(ticker.MultipleLocator(1))  # Ticks every 1 units

# plot with grid and Title
# plt.grid(which='major')
# plt.title("Rotating Vectors: i' is Blue and j' is Red: Original Vector is Green, Rotated Vector is Purple")

# indicate the value for all the vectors
plt.text(x, y, f"({"i':("}{x}, {y})", fontsize=10, ha='right')
plt.text(z, w, f"({"j':("}{z}, {w})", fontsize=10, ha='right')
plt.text(v0x, v0y, f"({"V0:("}{v0x}, {v0y})", fontsize=10, ha='right')
plt.text(v1x, v1y, f"({"V1:("}{v1x}, {v1y})", fontsize=10, ha='right')
plt.show(block=False)
# plt.Show the plot in a new interactive window
```

Math Circle 18 Matix Multiplication Determinants 4-5-6 2/12/2025 11:45to 1:15

Cyrus, Marc, Rooz, Dar

We continued the Linear Algebra series based on 3 Blue 1 Brown series by Grant Sanderson. We watched the 4th, 5th and 6th videos, and as before I also did a few demonstrations for them as I felt it was needed and we discussed matrix multiplication and determinants. Then they did the exercises on the 4th worksheet that I had prepared for them based on what Grant had covered in these videos. They learned about matrix multiplication and how it relates to doing 2 successive transformations. They also saw how determinants represented areas related to the parallelogram, rectangle or square formed by the basis vectors and some of the properties of the determinants. In addition, they saw how determinants represented volumes in 3-dimensional space. I am up to lecture 21 out of 35 lectures from the MIT linear algebra course by professor Strang and hunt for good stuff for the kids. I found one about how to analytically calculate the Area of triangles using determinants. that I will give them as a challenge next week.

Everyone did well. I encouraged and pulled Rooz along, and he did quite well, and I believe he gets it. Cyrus' made a great remark about determinants in 3D when they are zero out of the blue which was great recognizing that determinate of a 3x3 matrix being zero could mean that the vectors were on a plane or on a plane as well as all being zero.

At the end I projected what I had done in Python related to their exercise and displayed it on the TV.< I am attaching the code that Mihai can put on a computer and have Marc play with it.>

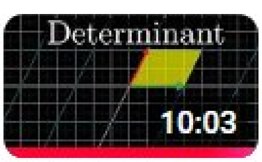
It was a good and very long session; we started when they were eating, and I am glad that they stayed with it. I feel they are getting a good feel for the basics of Linear Algebra.

4 

Matrix multiplication as composition | Chapter 4,...
3Blue1Brown

5 

Three-dimensional linear transformations | Chapter 5,...
3Blue1Brown

6 

The determinant | Chapter 6, Essence of linear algebra
3Blue1Brown

Name: Cyrus

Linear Algebra Problems (4)

Date: 2/12/25

1. Matrix $B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ defines a translation.
Draw & mark the basis vectors $\hat{\imath}_B$ and $\hat{\jmath}_B$ for this matrix.

2. Matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ defines another translation.
Draw and mark the basis vectors $\hat{\imath}_A$ and $\hat{\jmath}_B$ for this matrix.

3. Matrix $M = AB$. Find M.

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 4 & 2 \end{bmatrix}$$

4. For Matrix M draw and mark the basis vectors $\hat{\imath}_M$ & $\hat{\jmath}_M$

5. Draw the paralelegram formed by $\hat{\imath}_B$ and $\hat{\jmath}_B$
   a) find the area of this parelelogram.
   b) calculate the Determinant of B
   c) Are thes two values the same?

6&7. Repeat problem 5 fr matrecies A and M

8. Verify that:

$$Det(A)\,Det(B) = Det(M)$$

* 9. Find $N = BA$ $\quad N = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 5 & 3 \end{bmatrix}$

No Is $N = M$?
Yes Is $Det(N) = Det(M)$?

Cyrus 2/12/25

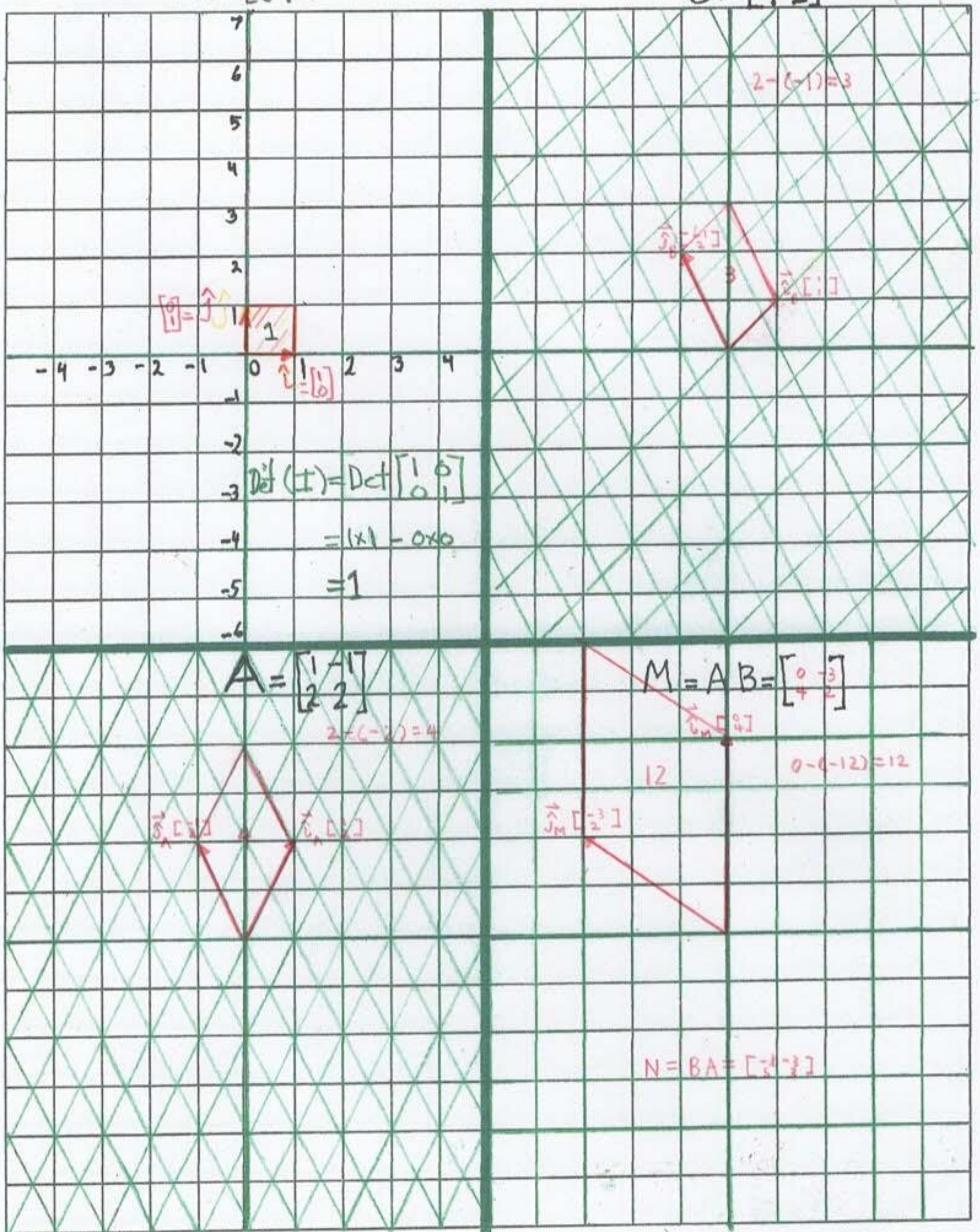
hand2mind

$I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

1cm METRIC GRAPH PAPER

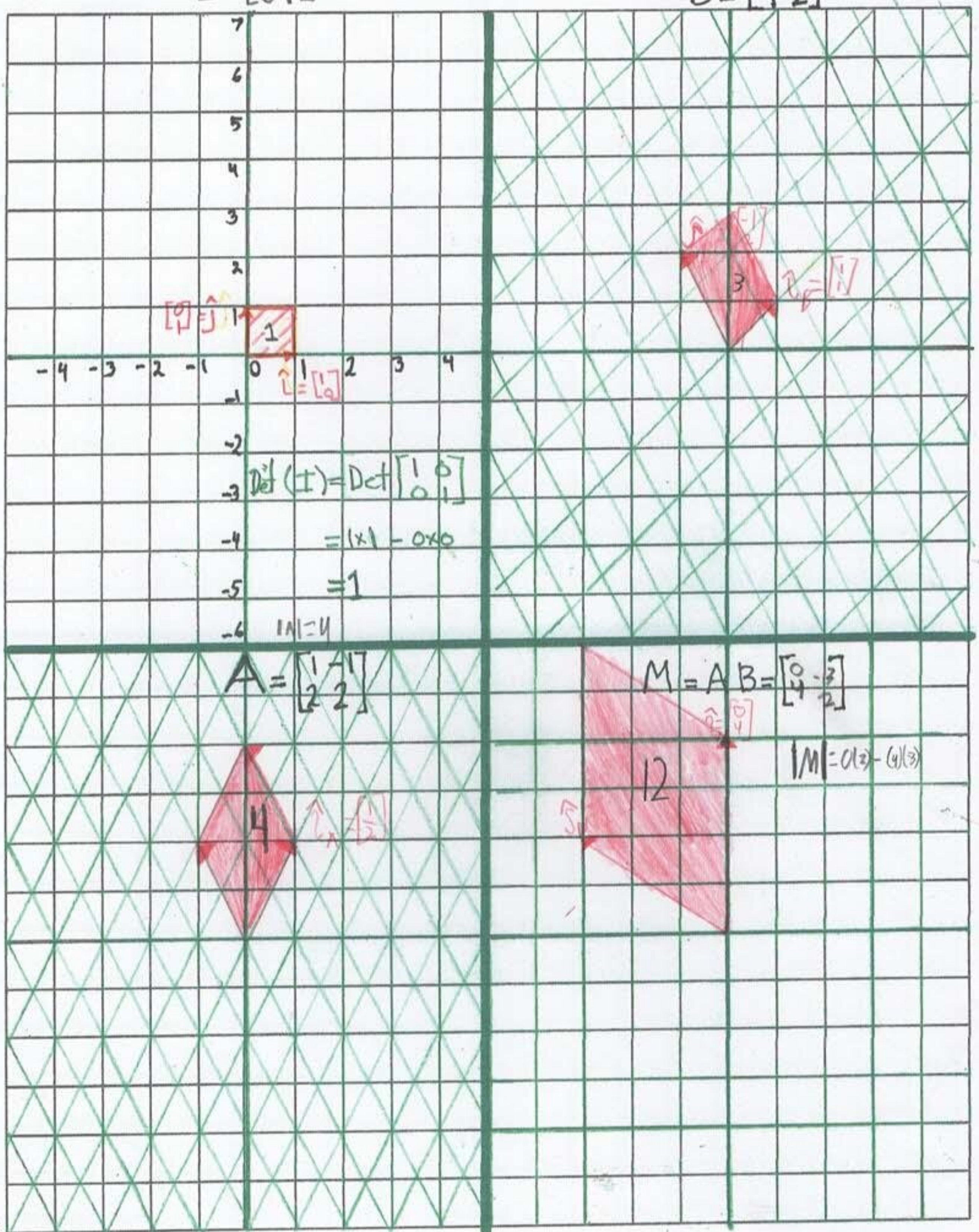
$B=\begin{bmatrix}1&-1\\1&2\end{bmatrix}$

4692



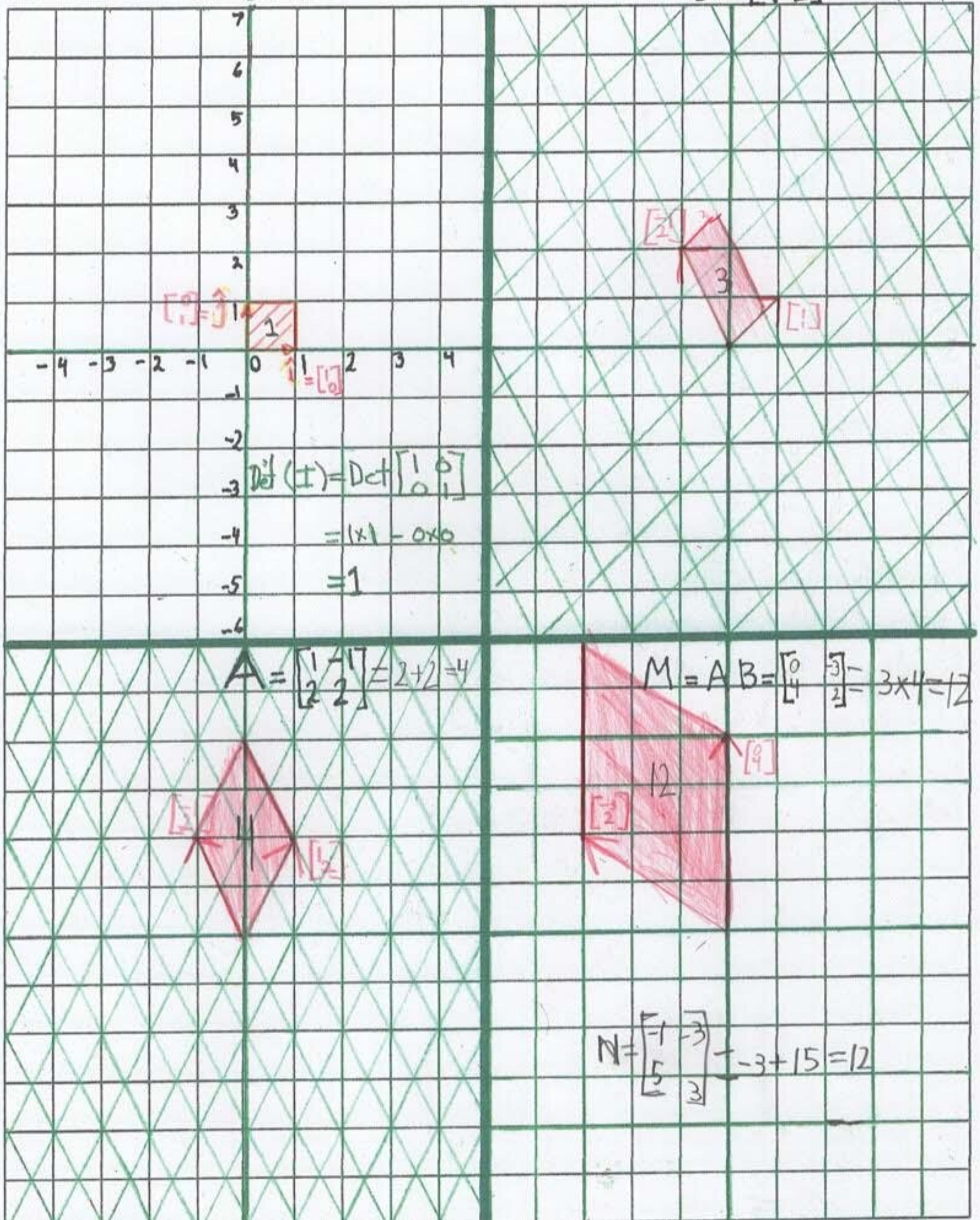
H474071005

hand2mind $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **1cm METRIC GRAPH PAPER** $B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ $|B| = 2+1$ 4692



H474071005

hand2mind $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ R007 1cm METRIC GRAPH PAPER $B = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = 1 \times 2 - 1 \times -1 = 3$ 4892



$Det(I) = Det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= 1 \times 1 - 0 \times 0$

$= 1$

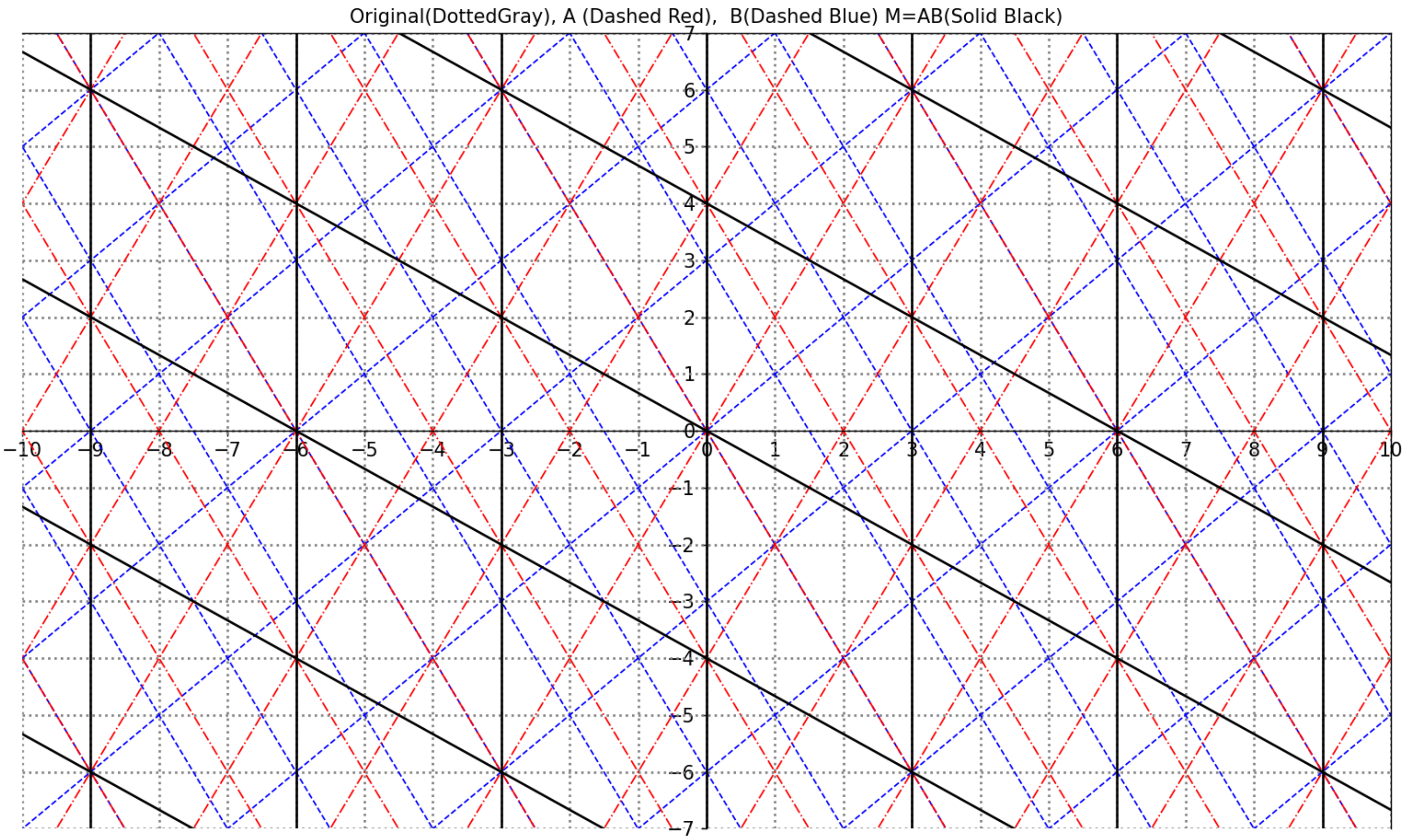
$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = 2 + 2 = 4$

$M = AB = \begin{bmatrix} 0 & -3 \\ 4 & 2 \end{bmatrix} = 3 \times 4 = 12$

$N = \begin{bmatrix} -1 & -3 \\ 5 & 3 \end{bmatrix} = -3 + 15 = 12$

H474071005

Python code for Shifting the Axis with Matrix Multiplication




```
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
import matplotlib.ticker as ticker
matplotlib.use("TkAgg")  # Force using Tkinter for separate window

# ******************************************************************************************
# ************************          Input Values          ******************************
# **************          define Matrecies A and B and M = ABsform          ******************
# ******************************************************************************************
B = np.array([[1,1], [-1,2]] ) # Row Vectors for Tranlation needs to be Transposed
Bij = np.transpose(B)         # column vectors for the Translation
A = np.array([[1,2], [-1,2]] ) # Row Vectors for Tranlation needs to be Transposed
Aij = np.transpose(A)         # column vectors for the Translation
M   = np.dot(B, A)            # Produces row vectors needs to be transposed
Mij = np.transpose(M)
# ***************************************************************************************************
# ***************************************************************************************************

# Function to create grid points ***********************************************************
def create_grid(x_range, y_range, step=1):
   x = np.arange(x_range[0], x_range[1] + step, step)
   y = np.arange(y_range[0], y_range[1] + step, step)
   grid_x, grid_y = np.meshgrid(x, y)
   return grid_x, grid_y

# Function to apply the Translation matrix B ************************************************
def rotate_pointsB(x, y):
   R = np.array(Bij)  # Rotation matrix
   xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
   return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# Function to apply the Translation matrix A *************************************************
```

```
def rotate_pointsA(x, y):
    R = np.array(Aij)  # Rotation matrix
    xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
    return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# Function to apply the Translation matrix M ***********************************************
def rotate_pointsM(x, y):
    R = np.array(Mij)  # Rotation matrix
    xy_rotated = np.dot(R, np.array([x.flatten(), y.flatten()]))  # Apply rotation
    return xy_rotated[0].reshape(x.shape), xy_rotated[1].reshape(y.shape)  # Reshape back to grid

# PLot size specification *******************************************************************
fig, ax = plt.subplots(figsize=(12, 12))

# Create a grid ******************************************************************************
grid_x, grid_y = create_grid((-10,10), (-7, 7), step=1)

# Draw original grid ************************************************************************
for i in range(grid_x.shape[0]):
    ax.plot(grid_x[i, :], grid_y[i, :], 'gray', linestyle='dotted')  # Horizontal lines
for j in range(grid_x.shape[1]):
    ax.plot(grid_x[:, j], grid_y[:, j], 'gray', linestyle='dotted')  # Vertical lines

# Translate the grid using Matrix B **********************************************************
rotated_x, rotated_y = rotate_pointsB(grid_x, grid_y)
# Draw Translated grid for B
for i in range(rotated_x.shape[0]):
    ax.plot(rotated_x[i, :], rotated_y[i, :], 'b', linestyle='dashed', linewidth="1")  # Horizontal lines
for j in range(rotated_x.shape[1]):
    ax.plot(rotated_x[:, j], rotated_y[:, j], 'b', linestyle='dashed', linewidth="1")  # Vertical lines

# Translate the grid using Matrix A **********************************************************
rotated_x, rotated_y = rotate_pointsA(grid_x, grid_y)
# Draw rotated grid for A
for i in range(rotated_x.shape[0]):
    ax.plot(rotated_x[i, :], rotated_y[i, :], 'r', linestyle='-.', linewidth="1")  # Horizontal lines
for j in range(rotated_x.shape[1]):
    ax.plot(rotated_x[:, j], rotated_y[:, j], 'r', linestyle='-.', linewidth="1")  # Vertical lines

# Translate the grid using Matrix M **********************************************************
rotated_x, rotated_y = rotate_pointsM(grid_x, grid_y)
# Draw rotated grid for M
for i in range(rotated_x.shape[0]):
    ax.plot(rotated_x[i, :], rotated_y[i, :], 'black', linestyle='-')  # Horizontal lines
for j in range(rotated_x.shape[1]):
    ax.plot(rotated_x[:, j], rotated_y[:, j], 'black', linestyle='-')  # Vertical lines

# Axis limit values,names and tick marks *****************************************************
#ax.set_xlabel("X-axis",fontsize=14, fontweight='bold')
#ax.set_ylabel("Y-axis",fontsize=14, fontweight='bold')
ax.spines['left'].set_position('zero')  # Y-axis at x=0
ax.spines['bottom'].set_position('zero')  # X-axis at y=0
plt.xlim(-10, 10)  # Set x-axis limits from 0 to 10
plt.ylim(-7, 7)  # Set y-axis limits from 0 to 60
ax.xaxis.set_major_locator(ticker.MultipleLocator(1))  # Ticks every 1 units
ax.yaxis.set_major_locator(ticker.MultipleLocator(1))  # Ticks every 1 units
ax.tick_params(axis='x', labelsize=12, labelcolor='black')  # Adjust size and color
ax.tick_params(axis='y', labelsize=12, labelcolor='black')  # Adjust size and color

# Chart Title ********************************************************************************
ax.set_title("Original(DottedGray), A (Dashed Red),  B(Dashed Blue) M=AB(Solid Black)")

# Plot everything ****************************************************************************
plt.show(block=False)
```

Math Circle 19 Inverse Matrices – Column and Row Space 2/12/2025 11:50 to 1:15

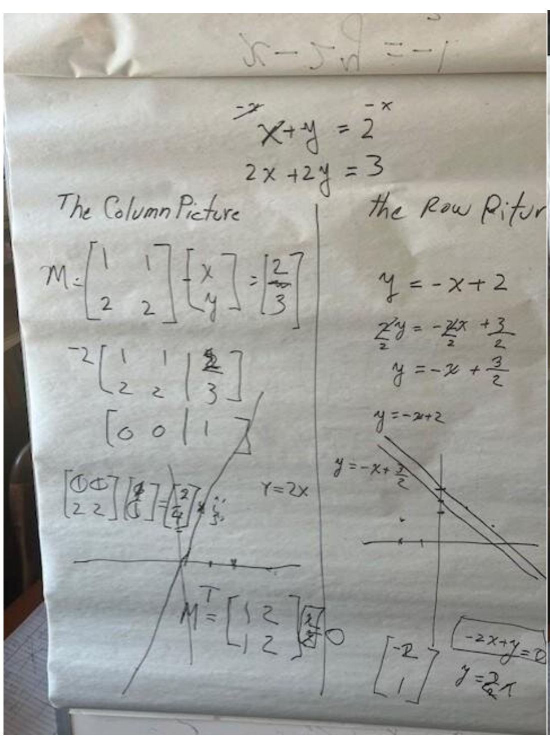
Cyrus, Marc, Rooz, Dar

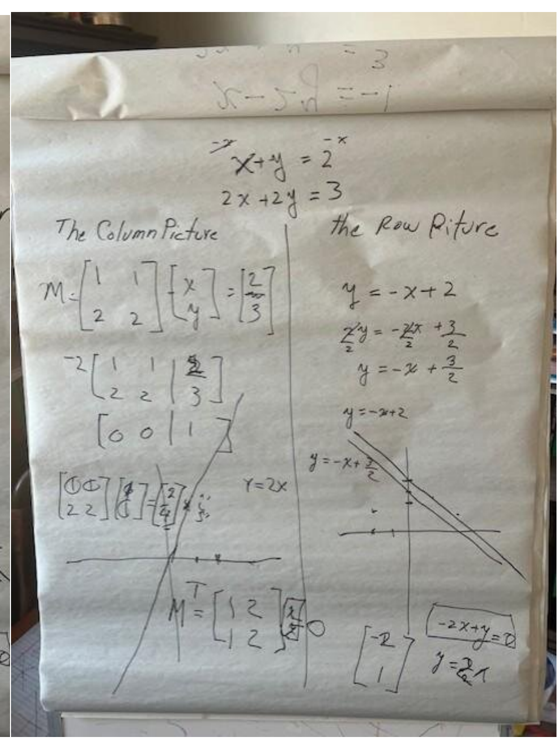
We continued the [Linear Algebra series based on 3 Blue 1 Brown series] by Grant Sanderson. We watched the 7thvideo. This time I had prepared a college type lecture for them which prepared them to do the exercise. The video dealt with Inverse matrix, Rank, Column Space and Null Space. I presented the algebraic and linear algebra pictures for solving systems of linear equations with two variables. I also covered the elimination method in the column picture. the determinants. In addition. I am up to lecture 29 out of 35 lectures from the MIT linear algebra course by professor Strang and hunt for good stuff for the kids;=> I found one about how to analytically calculate the Area of triangles using determinants. that I will give them as a challenge next week (I left this for next week's circle).

Everyone did well. Cyrus and Marc worked together as Marc had a finger cast and could not write. It really worked well as they were forced to talk to each other and have Cyrus write the answers after they were done discussing (no rush allowed;=). As always, I worked with Rooz and we made good progress even though a lot was knew for him like the equation of the line. He really make a great effort to tag along and it is great for him. I will review some of these concepts with him again separately.

They responded well to my college style lecture which I had worked on a lot.

It was a very good and very long session. They did not complain and were not even aware that they had put in nearly one how and twenty minute! I did get them some vegetable to eat at the start. I continue to feel they are getting a good feel for the basics of Linear Algebra.





Name: ROOZ

Date: ______

$x + y = 3$
$-x + y = 1$

Linear Algebra Problems (5)

The Column Picture

1. Find the Matrix M that represents this transformation:

$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}$

$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

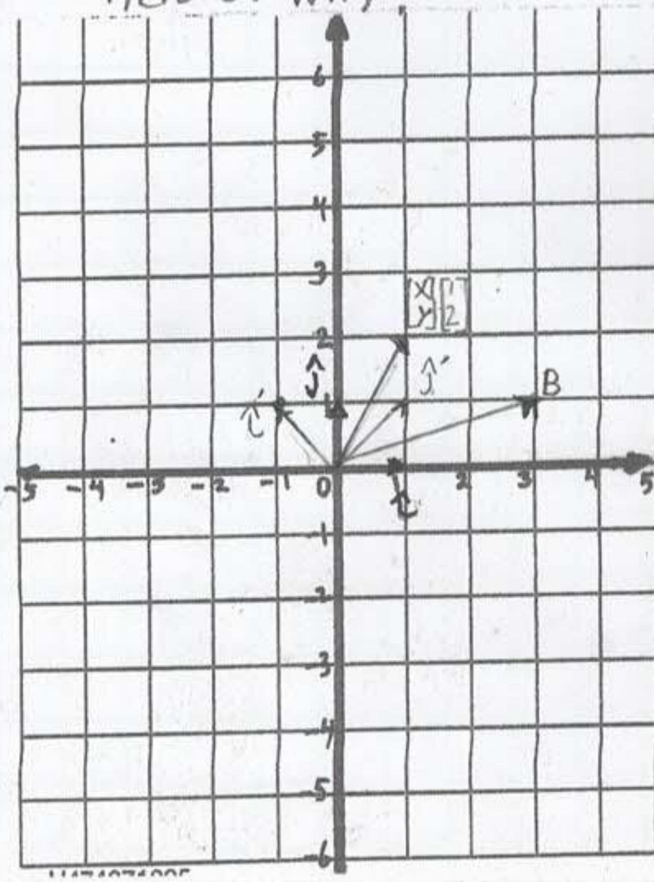
2. Using Elimination find $\begin{bmatrix} x \\ y \end{bmatrix}$ and draw $\hat{i}', \hat{j}'$ and vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$

$\left[M \,\middle|\, \begin{matrix} 3 \\ 1 \end{matrix}\right] = \left[\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix} \,\middle|\, \begin{matrix} 3 \\ 1 \end{matrix}\right]$

$x + 2 = 3 \quad = \left[\begin{matrix} 1 & 1 \\ 0 & 2 \end{matrix} \,\middle|\, \begin{matrix} 3 \\ 4 \end{matrix}\right] \quad 2y = 4, \; y = 2$

$x = 1 \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. Is this the same as $\begin{bmatrix} x \\ y \end{bmatrix}$ in Row Picture. Why?



The Row Picture

1. Write both equations in the form

$y = mx + a$

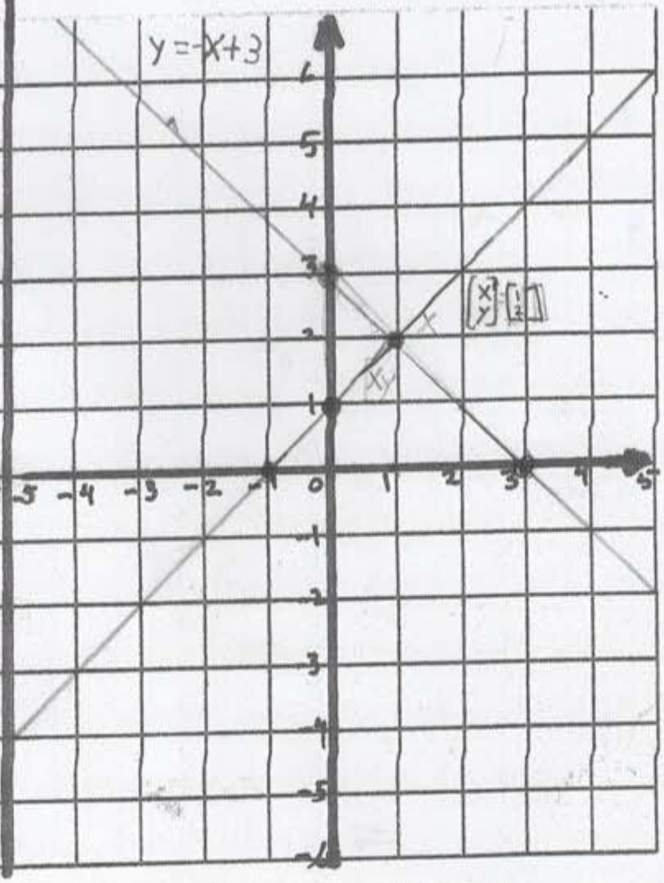
(m is slope, a is Intercept)

$y = -x + 3 \qquad -x + 3 = x + 1$

$y = x + 1 \qquad y = 2 \qquad 2 = 2x \quad x = 1$

2. Draw both lines below and find their intersection:

$\begin{bmatrix} x \\ y \end{bmatrix} =$



Name: ROOZ

Date: ______

$$2x + y = c$$
$$4x + 2y = d$$

Linear Algebra Problems (6)

1. For $c=2$ & $d=2$ write and draw $\hat{i}'$ and $\hat{j}'$.

$$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

2. Can you find $\begin{bmatrix} x \\ y \end{bmatrix}$? Why?

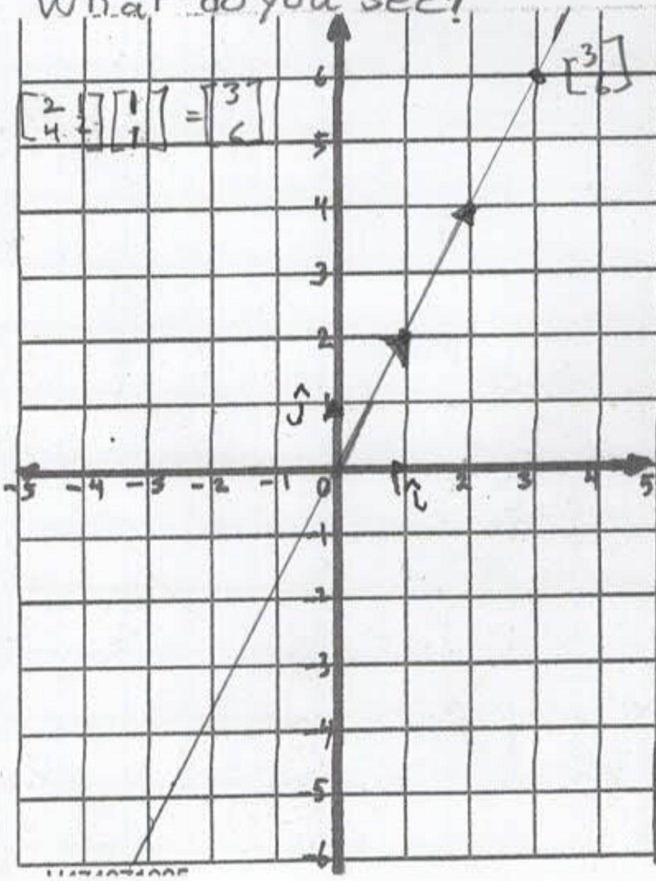
$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad -2 \left[\begin{array}{cc|c} 2 & 1 & 2 \\ 0 & 0 & -2 \end{array}\right]$$

3. For $c=4$ & $d=8$ Find $\begin{bmatrix} x \\ y \end{bmatrix}$. How many can you find?

$$\left[\begin{array}{cc|c} 2 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

4. Find the Null space of $M^T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ and draw it $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ What do you see?

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$



1. For $c=2$ & $d=2$ write and draw these lines:

$4x+2y=2$

$2x=2$
$x=1$

$y = -2x + 2$

$y = -2x + 1$

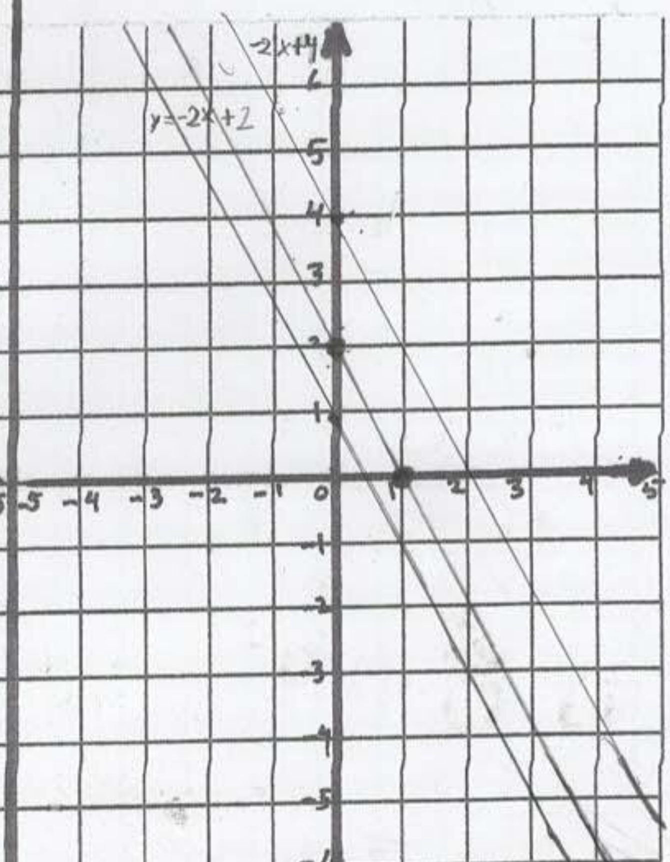
2. Do these lines intersect?

3. For $c=4$ & $d=8$ write and draw thes new lines:

$y = -2x + 4$

$y = -2x + 4$

4. What do you see?



Name: Cyrus & Marc

Date: 2/26/25

$$\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}$$

$$x + y = 3$$
$$-x + y = 1$$

Linear Algebra Problems (6)

The Column Picture

1. Find the Matrix M that represents this transformation:

$$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}$$

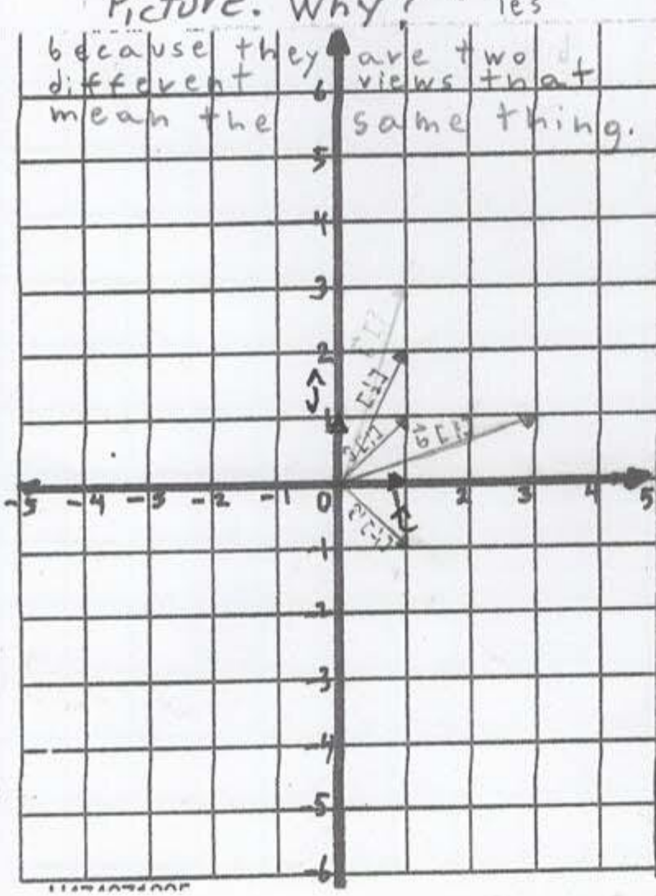
2. Using Elimination find $\begin{bmatrix} x \\ y \end{bmatrix}$ and draw $\hat{\imath}$, $\hat{\jmath}$ and vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\left[M \,\middle|\, \begin{matrix} 3 \\ 1 \end{matrix}\right] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad y = 2 \quad x = 1$$

3. Is this the same as $\begin{bmatrix} x \\ y \end{bmatrix}$ in Row Picture. Why? Yes

because they are two different views that mean the same thing.



The Row Picture

1. Write both equations in the form

$$y = mx + a$$

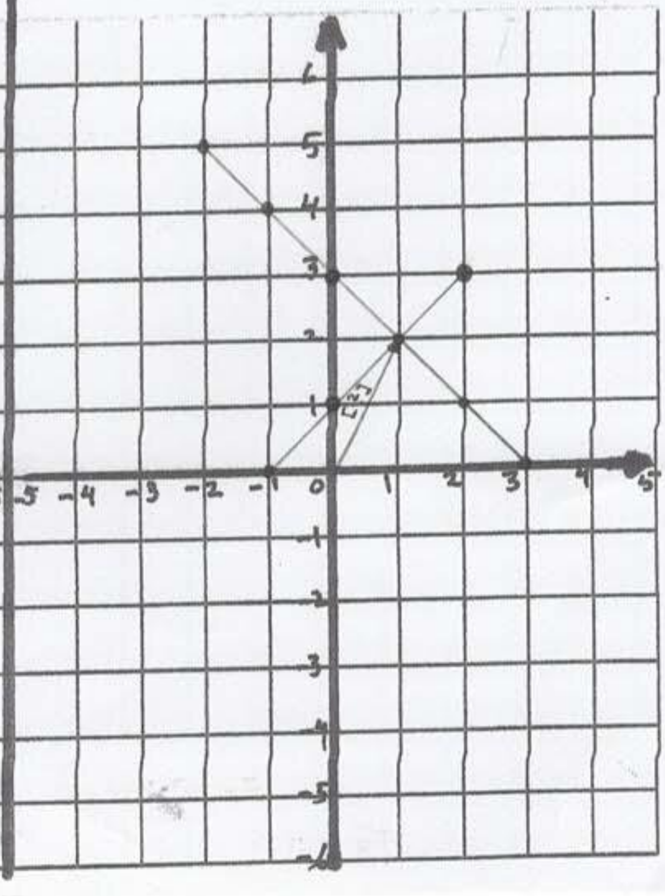
(m is slope, a is Intercept)

$y = -x+3$

$y = x+1$

2. Draw both lines below and find their intersection:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Name: Cyrus & Marc

Date: 2/26/25

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$2x + y = c$$
$$4x + 2y = d$$

Linear Algebra Problems (6)

1. For $c=2$ & $d=2$ write and draw $\hat{i}'$ and $\hat{j}'$.

$$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

2. Can you find $\begin{bmatrix} x \\ y \end{bmatrix}$? Why?

No because they are parralell

3. For $c=4$ & $d=8$ Find $\begin{bmatrix} x \\ y \end{bmatrix}$.

How many can you find?

infinety

4. Find the Nullspace of $M^T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ and draw it $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

What do you see?

1. For $c=2$ & $d=2$ write and draw these lines

$y = -2x+2$

$y = -2x+1$

2. Do these lines intersect?

No

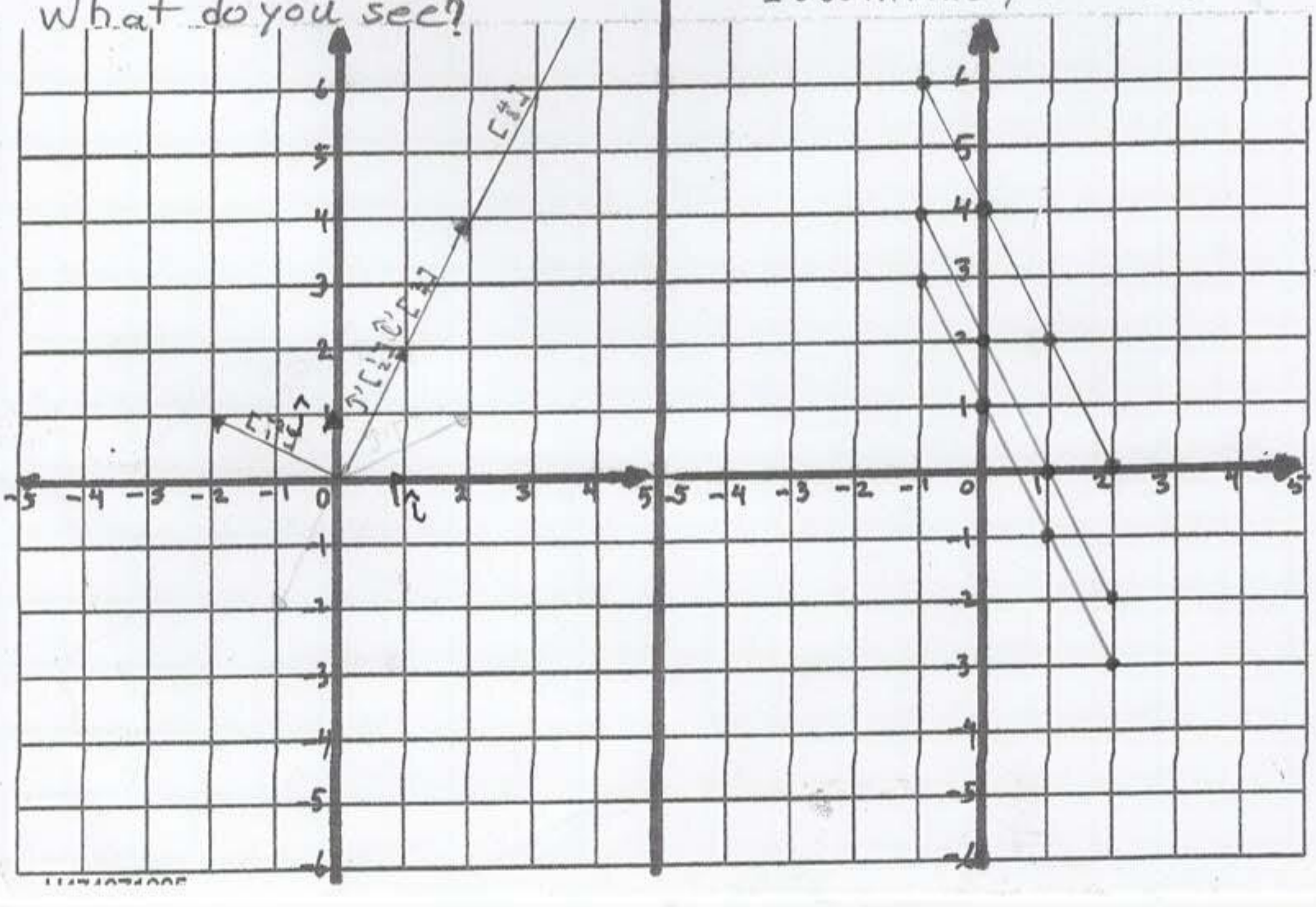
3. For $c=4$ & $d=8$ write and draw thes new lines:

$y = -2x+4$

$y = -2x+4$

4. What do you see?

I see infinety answers.



Name: ______

Date: ______

$$x + y = 3$$
$$-x + y = 1$$

Linear Algebra Problems (5)

The Column Picture

1. Find the Matrix M that represents this transformation:

$$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}$$

2. Using Elimination find $\begin{bmatrix} x \\ y \end{bmatrix}$ and draw $\hat{i}$, $\hat{j}$ and vectors $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\left[M \,\middle|\, \begin{matrix} 3 \\ 1 \end{matrix} \right] =$$

3. Is this the same as $\begin{bmatrix} x \\ y \end{bmatrix}$ in Row Picture. Why?

The Row Picture

1. Write both equations in the form

$$y = mx + a$$

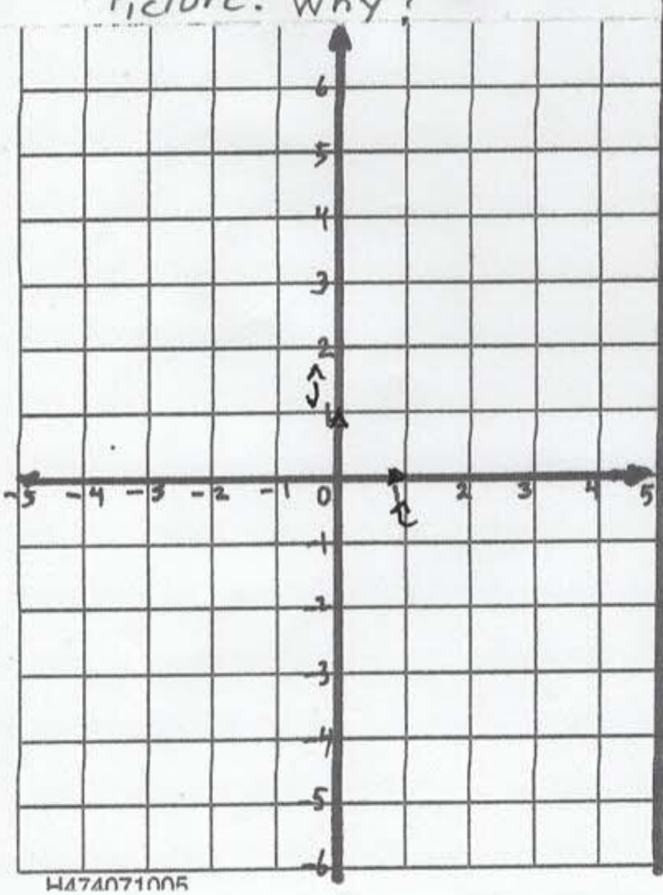
(m is slope, a is Intercept)

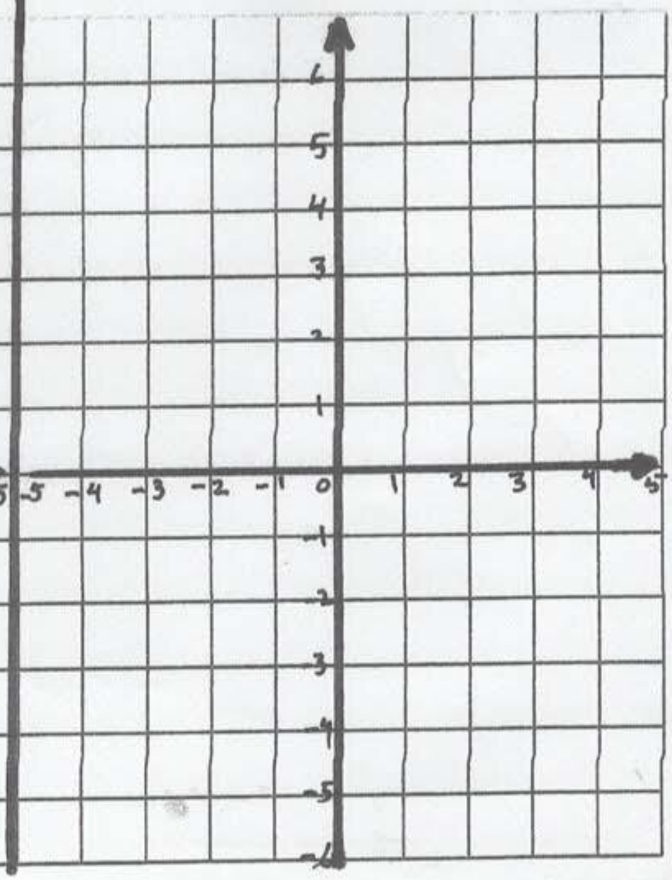
$y =$

$y =$

2. Draw both lines below and find their intersection:

$$\begin{bmatrix} x \\ y \end{bmatrix} =$$





H474071005

Name: ______

Date: ______

$2x + y = c$
$4x + 2y = d$

Linear Algebra Problems (6)

1. For $c = 2$ & $d = 2$ write and draw $\hat{\imath}'$ and $\hat{\jmath}'$.

$$[M]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

2. Can you find $\begin{bmatrix} x \\ y \end{bmatrix}$? Why?

3. For $c = 4$ & $d = 8$ Find $\begin{bmatrix} x \\ y \end{bmatrix}$. How many can you find?

4. Find the Null space of $M^T = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ and draw it $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ What do you see?

1. For $c = 2$ & $d = 2$ write and draw these lines

$y =$

$y =$

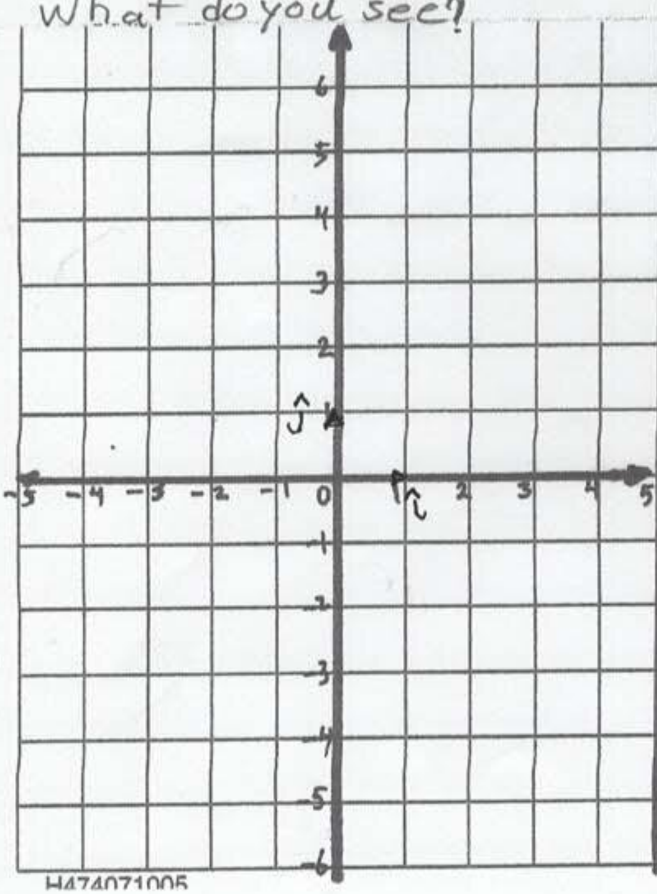
2. Do these lines intersect?

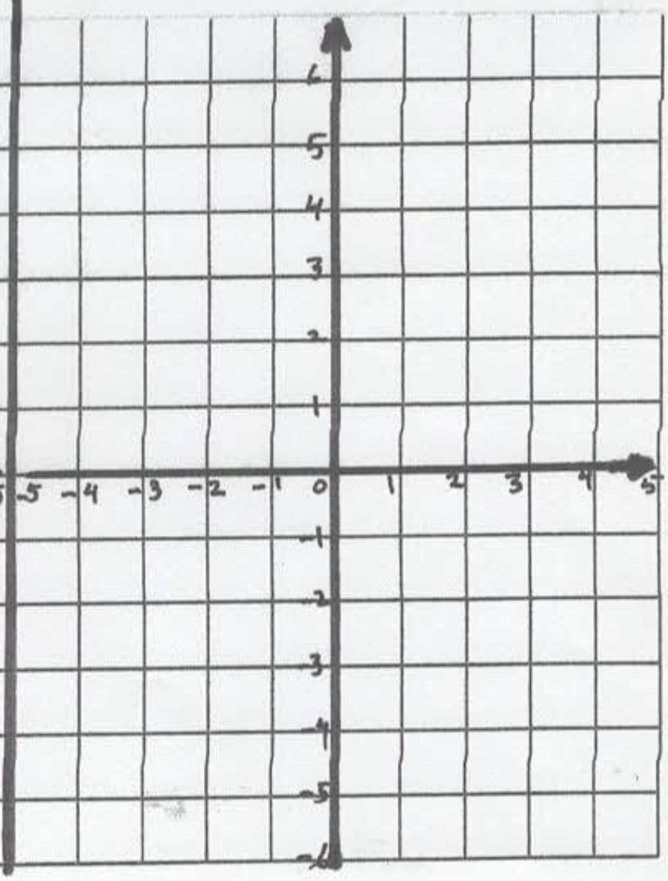
3. For $c = 4$ & $d = 8$ write and draw thes new lines:

$y =$

$y =$

4. What do you see?





Math Circle 20 Linear Algebra 5 3/05/2025 1:55 to 2:20

Cyrus, Marc, Rooz, Dar

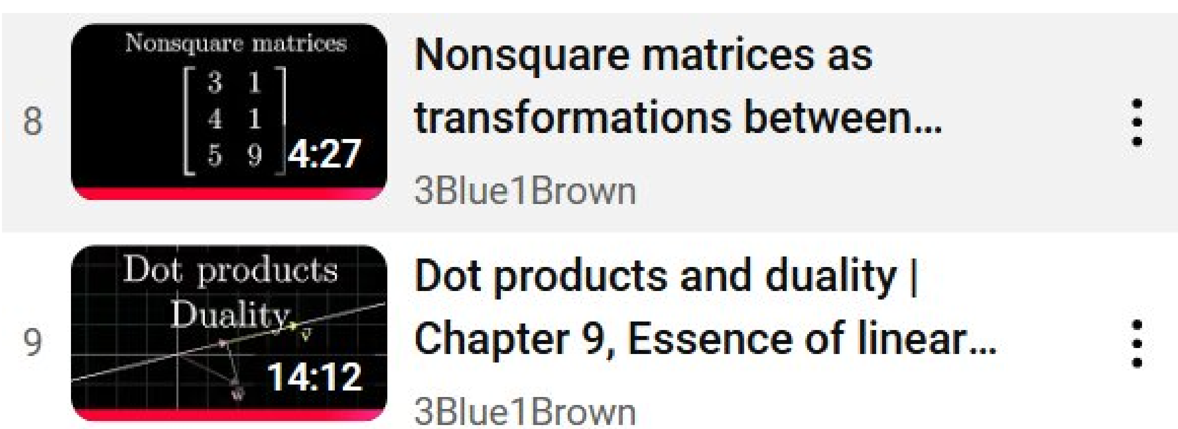
We continued the Linear Algebra series based on 3 Blue 1 Brown series by Grant Sanderson. We watched the 8th and 9th video. This time I had also prepared a college type lecture for them discussing the problem of the meaning of cross product and projection of one vector on the other one the and the incredible fact that according to these two definitions you could multiply 3 irrational numbers and get a digit as a result! I have watched the videos many times myself and finally I had a good understanding of this duality which is intriguing and not easy to grasp. I solved the problem in my sleep last night after thinking about it for nearly two weeks.

I reviewed a little Trig for them and in particular for Rooz before my presentation. The presentation was tricky, but I believe Cyrus and Marc essentially understood what I was describing and Rooz probably just got a whiff of it. Nevertheless, I think they got a chance to see how an original problem arises and they appreciated my enthusiasm about the problem and the solution. Then we watched the two videos, and they worked on the Projection and Dot Product work sheet that I had prepared for them. Finally, they completed the worksheet on calculating the area of triangles using determinants.

I am up to lecture 22 out of 35 lectures from the MIT linear algebra course by professor Strang and hope to finish them soon and go back and study the Book.

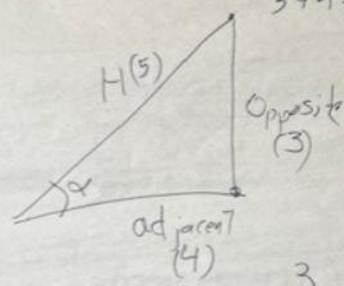
Everyone did well. Cyrus and Marc talked together some while doing their work. As always, I worked with Rooz, and we made good progress even though a lot was more difficult for him. He really makes a great effort to tag along, and it is great to see him do that. I will review some of these concepts with him again separately.

It was again a very good and very long session. They did not even complain and were not even aware that they had put in nearly one how and twenty minutes again! I did get them some vegetables to eat at the start and gave Rooz some rice in the middle. I continue to feel they are getting a good feel for the basics of Linear Algebra. I am also getting a good feel.



SOH
COH
TOA

$3^2+4^2=5^2$



$\sin\alpha = \frac{Op}{H} = \frac{3}{5} = .6$

$\cos\alpha = \frac{ad}{H} = \frac{4}{5} = .8$

$\tan\alpha = \frac{Op}{ad} = \frac{\frac{Op}{H}}{\frac{ad}{H}} = \frac{\sin\alpha}{\cos\alpha}$

$= \frac{3}{4} = .75$

$\vec{V}\cdot\vec{U} = \vec{U}\cdot\vec{V} = [2\ 4]\begin{bmatrix}5\\1\end{bmatrix} = 2\times5+4\times1$

$\vec{V}\cdot\vec{U} = \|\vec{U}\|\times\|\vec{V}\| = 14$

$= \|\vec{V}\|\,\|\vec{U}\|\cos\alpha$

$\alpha = 52.125$

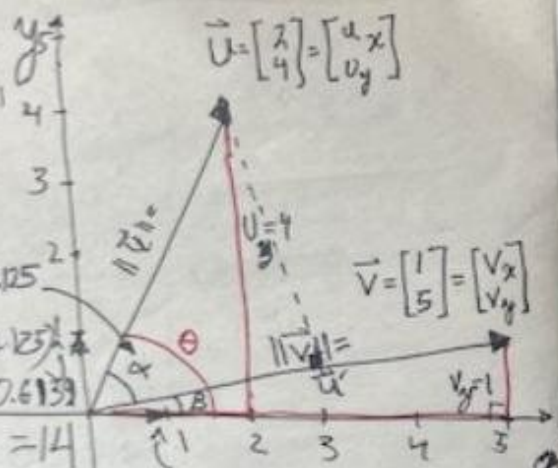
$= \sqrt{26} * \sqrt{20}\cos(52.125)$

$= 5.099\times4.472\times0.6139$

$= 14$

$\vec{U} = \begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}u_x\\u_y\end{bmatrix}$

$\vec{V} = \begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}V_x\\V_y\end{bmatrix}$



$U_x = 2$ $\quad V_x = 5$

$[U_x\ V_y]\begin{bmatrix}V_x\\V_y\end{bmatrix}$

$\boxed{U_x V_x + U_y V_y}$

$\boxed{\alpha = \theta - \beta}$

$\sin\beta = \frac{V_y}{\|\vec{V}\|} \quad \cos\beta = \frac{V_x}{\|\vec{V}\|}$

$\sin\theta = \frac{U_y}{\|\vec{U}\|} \quad \cos\theta = \frac{U_x}{\|\vec{U}\|}$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(\alpha) = \cos(\theta-\beta) = \cos\theta\cos(-\beta) - \sin\theta\sin(-\beta)$

$= \cos\theta\cos\beta + \sin\theta\sin\beta$

$= \frac{U_x V_x}{\|\vec{U}\|\,\|\vec{V}\|} + \frac{V_y U_y}{\|\vec{U}\|\,\|\vec{V}\|}$

$\cos(\alpha) = \frac{V_x U_x + V_y U_y}{\|\vec{U}\|\,\|\vec{V}\|}$

$\boxed{V_x U_x + V_y U_y = \|\vec{V}\|\,\|\vec{U}\|\cos\alpha}$

Q.E.D.

Name: Cyrus

Linear Algebra Problems (7)

Date: 3/5/25

1. Find and mark the value of the vector $\vec{A}, \vec{B}, \vec{C}, \vec{D}$.
   $\vec{A}\begin{bmatrix} 6 \\ 11 \end{bmatrix}$ $\vec{B}\begin{bmatrix} 9 \\ 2 \end{bmatrix}$ $\vec{C}\begin{bmatrix} -9 \\ -8 \end{bmatrix}$ $\vec{D}\begin{bmatrix} 10 \\ -4 \end{bmatrix}$

2. Find the matrecies M & N where:

$$M = \begin{bmatrix} \vec{A} & \vec{B} \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 11 & 2 \end{bmatrix} \quad N = \begin{bmatrix} \vec{C} & \vec{D} \end{bmatrix} = \begin{bmatrix} -9 & 10 \\ -8 & -4 \end{bmatrix}$$

* 3. Find the area of the triangles M & N

Area of $M = -43\frac{1}{2}$ Area of $N = 58$

4. Repeat thes steps for vectors $\vec{H}$ and $\vec{I}$ and

Find $Q = \begin{bmatrix} \vec{H} & \vec{I} \end{bmatrix} = \begin{bmatrix} -4 & -9 \\ 8 & 2 \end{bmatrix}$ and th Area of $Q = 32$

* 5. Find the area of the triangle P with vertecies EFQ.

Area of $P = 32$

** 6. For any triangle can you use the same method to find its Area?

If the vertecies are at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$

what is the Area interms of a, b, c, d

Area $= (ad - bc) \div 2$

Name:

Linear Algebra Problems (7)

Date:

1. Find and mark the value of the vector $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$.

2. Find the matrecies M & N where:

$$M = \begin{bmatrix} \vec{A} & \vec{B} \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 11 & 2 \end{bmatrix} \quad N = \begin{bmatrix} \vec{C} & \vec{D} \end{bmatrix} = \begin{bmatrix} -9 & 10 \\ -8 & -4 \end{bmatrix}$$

*3. Find the area of the triangles M & N

Area of M = -43.5 Area of N = 58

det M = 12 - 99 = -87 det N = 36 + 80 = 116

4. Repeat thes steps for vectors $\vec{H}$ and I and

Find $Q = \begin{bmatrix} \vec{H} & \vec{I} \end{bmatrix} = \begin{bmatrix} -4 & -9 \\ 8 & 2 \end{bmatrix}$ and th Area of Q = 32

det Q = -8 + 72 = 64

*5. Find the area of the triangle P with vertecies EFQ.

Area of P = 32

**6. For any triangle can you use the same method to find its Area?

If the vertecies are at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$

what is the Area interms of a, b, c, d

$$\text{Area} = \frac{ad - bc}{2} \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Name: Marc

Date:

Linear Algebra Problems (7)

1. Find and mark the value of the vector $\vec{A}, \vec{B}, \vec{C}, \vec{D}$.

2. Find the matrecies M & N where:

$$M = \begin{bmatrix} \vec{A} & \vec{B} \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 11 & 2 \end{bmatrix} \quad N = \begin{bmatrix} \vec{C} & \vec{D} \end{bmatrix} = \begin{bmatrix} -9 & 10 \\ -8 & -4 \end{bmatrix}$$

*3. Find the area of the Triangles M & N

Area of M = -43.5 Area of N = 58

4. Repeat thes steps for vectors $\vec{H}$ and $\vec{I}$ and

Find $Q = \begin{bmatrix} \vec{H} & \vec{I} \end{bmatrix} = \begin{bmatrix} -4 & -9 \\ 8 & 2 \end{bmatrix}$ and th Area of Q = 64

*5. Find the area of the triangle P with vertecies EFQ.

Area of P = 64

** 6. For any triangle can you use the same method to find its Area?

If the vertecies are at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$

what is the Area interms of a,b,c,d

$$\text{Area} = \frac{ad-bc}{2}$$

Name: Cyrus

Date: 3/5/25

Linear Algebra

8

Projeting on x-axis & y axis Dot product with $\hat{\imath}$ & $\hat{\jmath}$	Relating Projection with Dot Product	Projecting a vetor on another vector

1. Draw the vector $\vec{U} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
2. Project $\vec{U}$ on the x-axis
3. Calculate the dotproduct of $\hat{\imath}$ with $\vec{U}$

   $\hat{\imath} \cdot \vec{U}$ $\quad [1\ 0]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2$

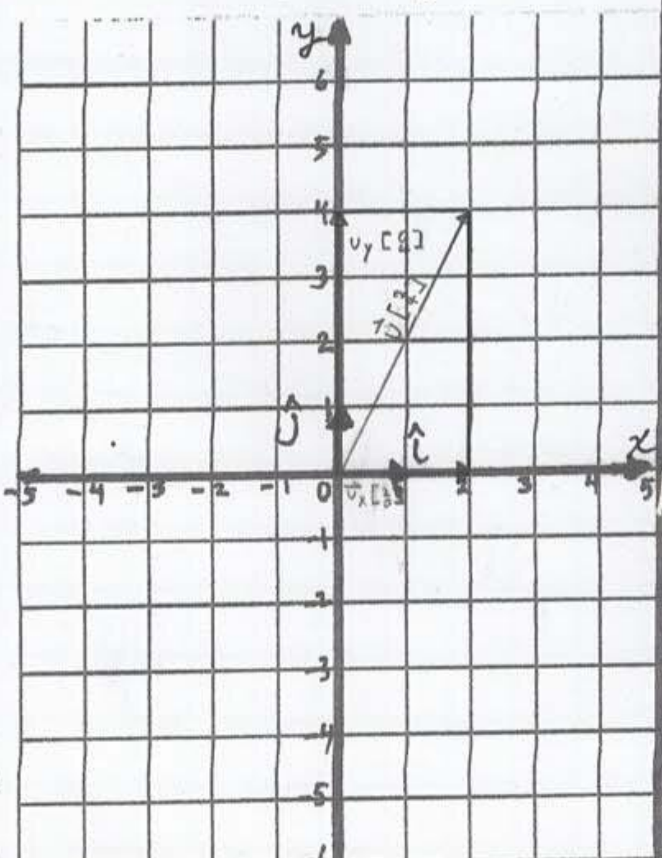
   Is this the length of the Projection of $\vec{U}$ on x-axis U_x? yes no
4. Calculate the dot product of $\hat{\jmath}$ with $\vec{U}$

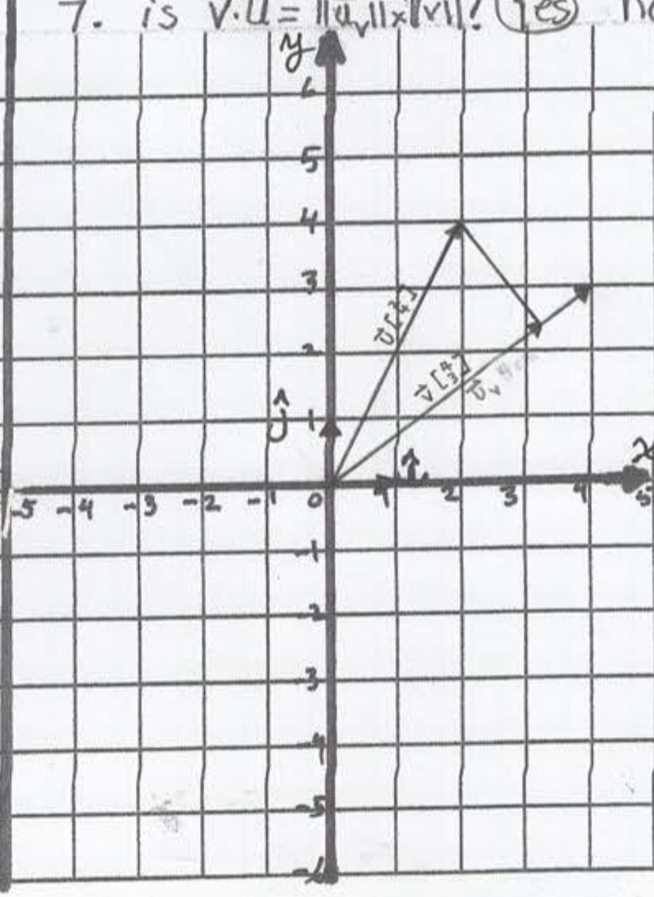
   $\hat{\jmath} \cdot \vec{U}$ $\quad [0\ 1]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 4$

   Is this the length of the Projection of $\vec{U}$ on the y-axis U_y? Yes no

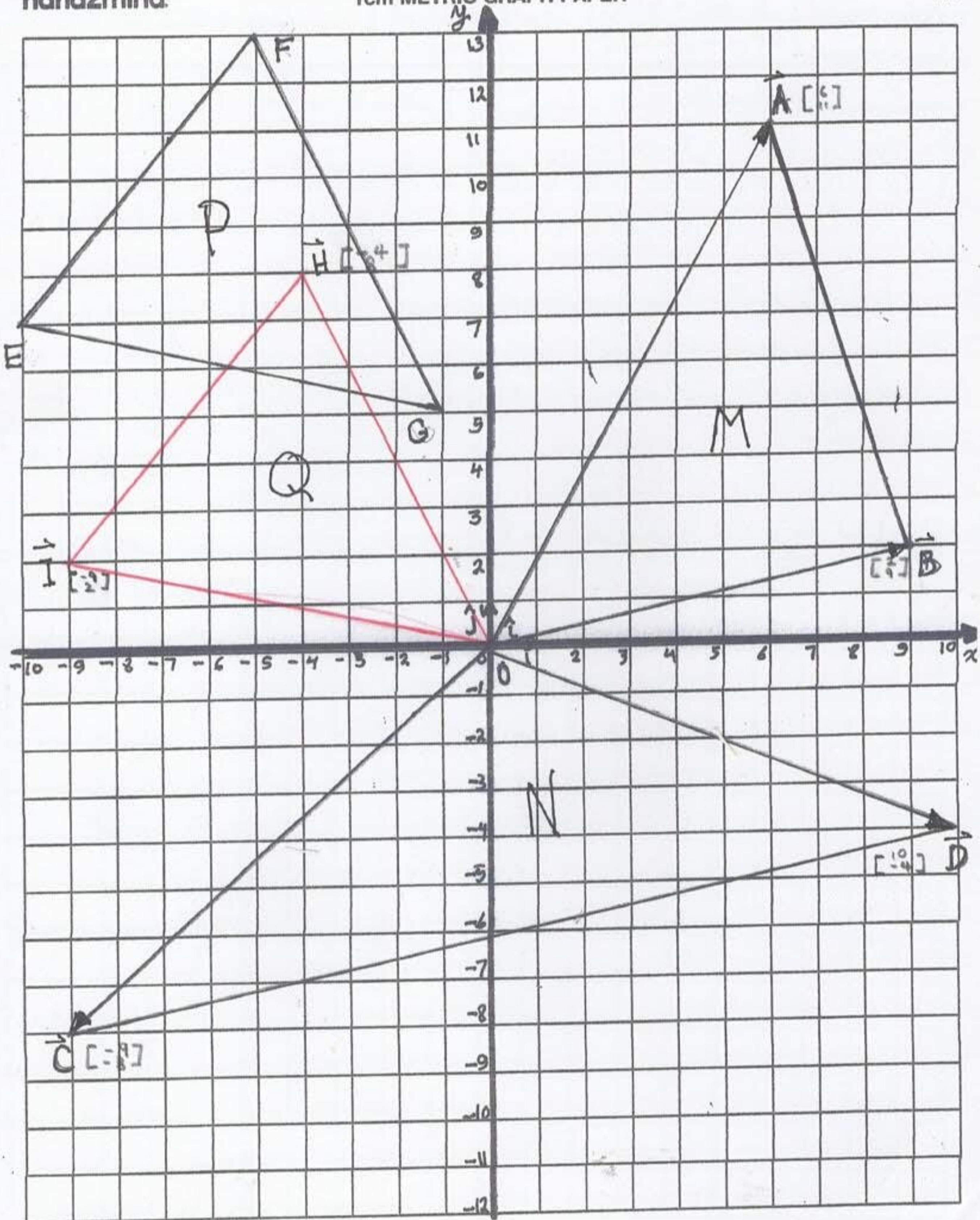
1. Draw the vectors $\vec{U} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ & $V = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$
2. Project $\vec{U}$ on $\vec{V}$ and mark it $\vec{U}_V$
3. Find the dot product of V and U

   $\vec{V} \cdot \vec{U} = [4\ 3]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20$
4. Find the length of $\vec{V}$ using the Pythagorean theorem. $\|\vec{V}\| = 5\text{cm}$
5. Measure the length of $\vec{U}_V$ with a ruler $\|\vec{U}_V\| = 4\text{cm}$
6. Calculate $\|\vec{U}_V\| \times \|\vec{V}\| = 20\text{ cm}$
7. is $\vec{V} \cdot \vec{U} = \|\vec{U}_V\| \times \|\vec{V}\|$? Yes no





hand2mind 1cm METRIC GRAPH PAPER 4692



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Name: Rooz
Date: ______
Linear Algebra 8

Projeting on x-axis & y axis Dot product with $\hat{i}$ & $\hat{j}$	Relating Projection with Dot Product	Projecting a vetor on another vector

1. Draw the vector $\vec{U} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
2. Project $\vec{U}$ on the x-axis
3. Calculate the dotproduct of $\hat{i}$ with $\vec{U}$

$\hat{i} \cdot \vec{U} \quad [1\ 0]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1\times2+0\times4 = 2$

Is this the length of the Projection of $\vec{U}$ on x-axis U_x? yes no

4. Calculate the dot product of $\hat{j}$ with $\vec{U}$

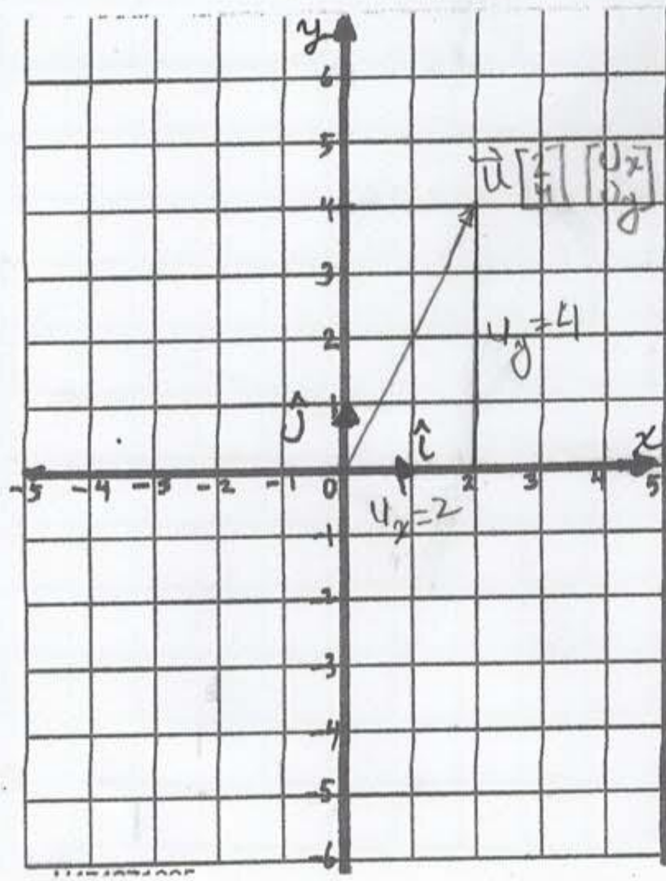
$\hat{j} \cdot \vec{U} \quad [0\ 1]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0\times2+1\times4 = 4$

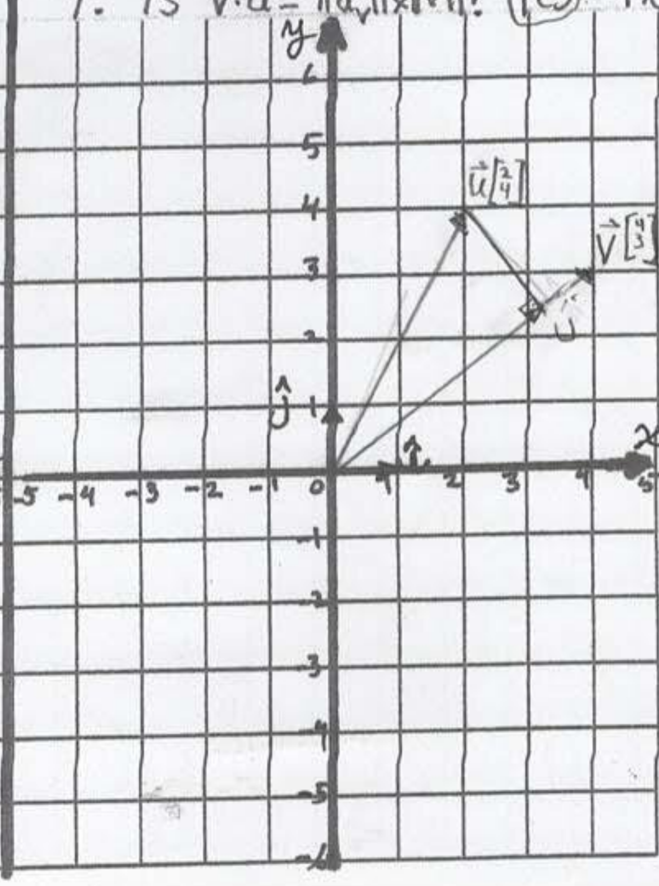
Is this the length of the Projection of $\vec{U}$ on the y-axis U_y? Yes no

1. Draw the vectors $\vec{U} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ & $V = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$
2. Project $\vec{U}$ on $\vec{V}$ and mark it $\vec{U}_V$
3. Find the dot product of V and U

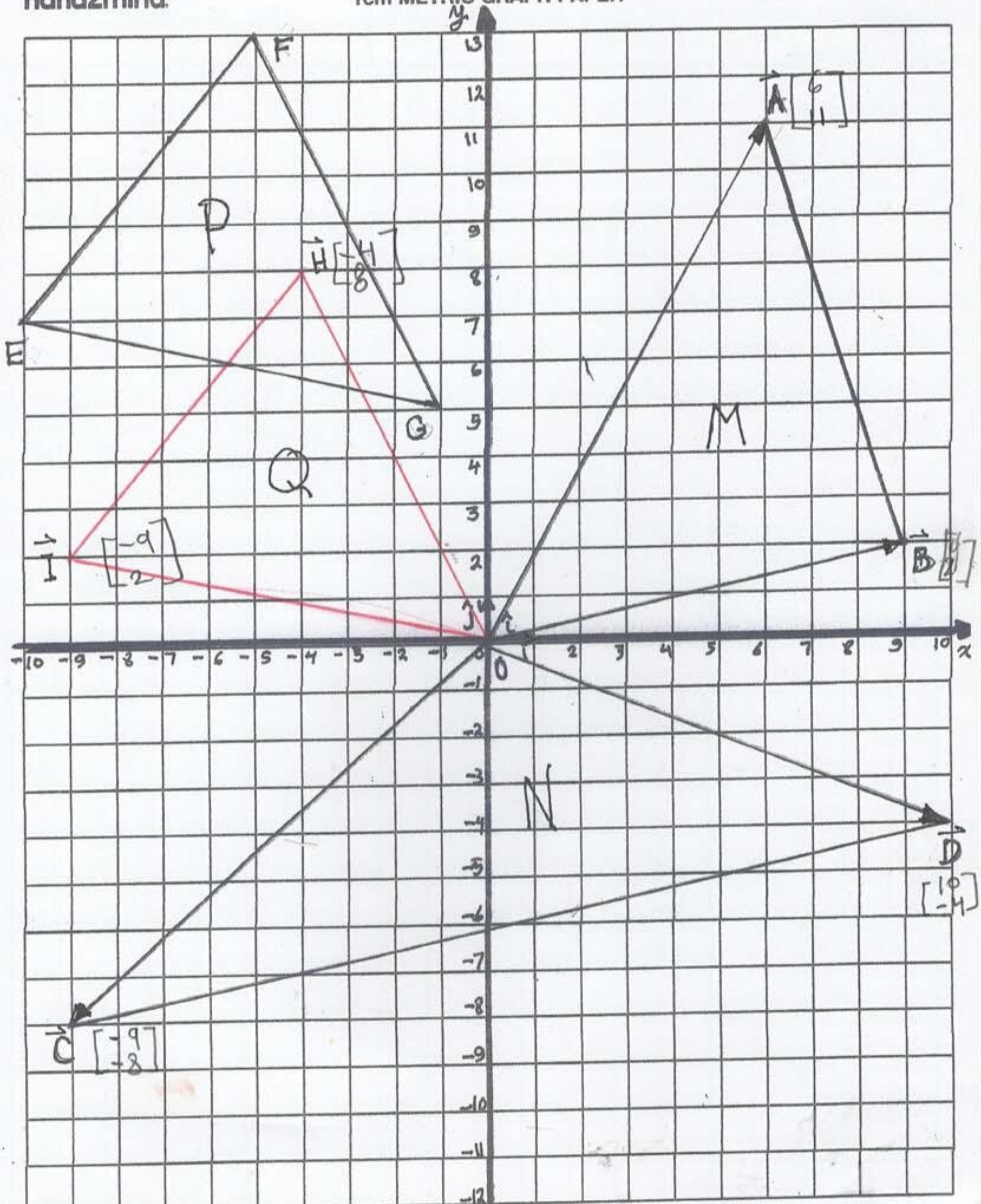
$\vec{V} \cdot \vec{U} = [4\ \ 3]\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20$

4. Find the length of $\vec{V}$ using the Pythagorean theorem. $\|\vec{V}\| = 5$
5. Measure the length of $\vec{U}_V$ with a ruler $\|\vec{U}_V\| = 4$.
6. Calculate $\|\vec{U}_V\| \times \|\vec{V}\| = 20$
7. is $\vec{V} \cdot \vec{U} = \|\vec{U}_V\| \times \|\vec{V}\|$? Yes no





hand2mind 1cm METRIC GRAPH PAPER 4692



Math Circle 21 Linear Algebra 6 3/19/2025 1:55 to 2:00

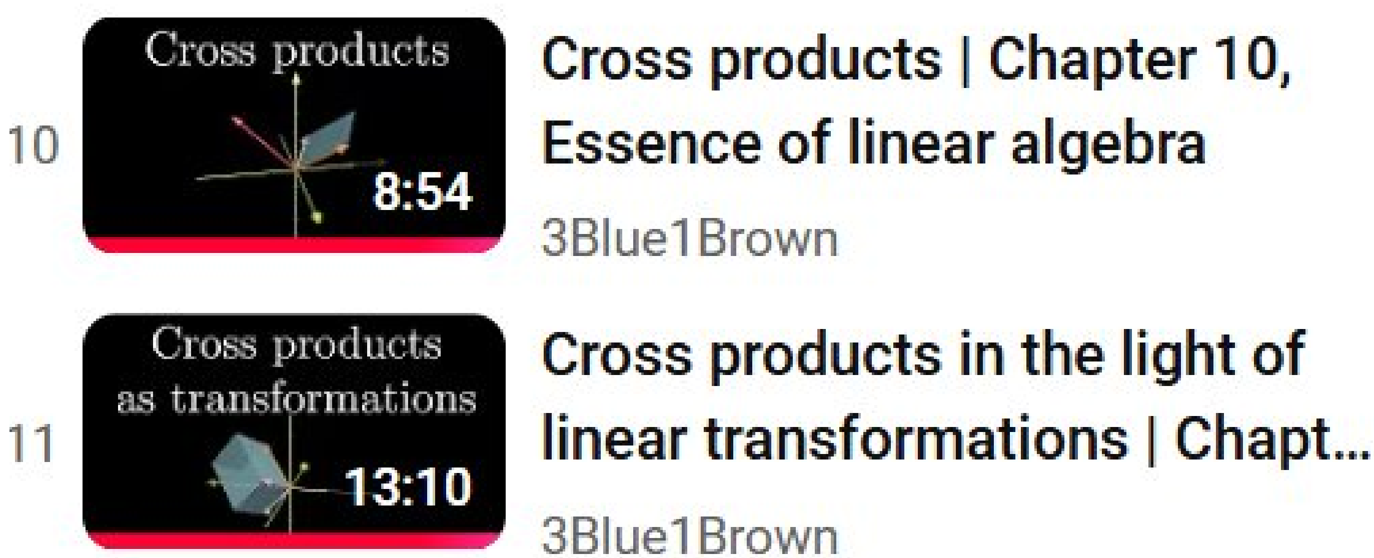
Cyrus, Marc, Rooz, Dar

We continued the [Linear Algebra series based on 3 Blue 1 Brown series] by Grant Sanderson. We watched the 10th and 11th videos. These videos are amazing! This time also I spent a lot of time mastering the cross product and preparing 3 sheets for them that I also did with them on board. They had to see two ways to find the volume of a parallelopiped and connect this to the meaning of the dot product to find the length of the cross product of 2 vectors. This involved 2 dualities which are really nuanced to get.

First, we watched the first video. Then I worked with them with the 2 work sheets. Next, we watched the second video and finally, we worked on the last worksheet, and I went over the concepts several times to make sure that they got the main connection between the cross product and the volume of an arbitrary parallelopiped made on the 2 original vectors. I tried to use as many numbers as possible instead of symbols to make the understanding easier.

Everyone did well. Cyrus and Marc, I think understood all the nuances and Rooz was at least at 50% so I think this went well as it was quite complicated for me to get and to compose stuff for them that would guide them to the true meaning of cross product of 2 vectors. I worked with Rooz, and we made good progress even though a lot was more difficult for him. He really made a great effort again to tag along, and it is great to see an 81/2 year old get concepts that are difficult even for college students <not mentioning PhDs in Math;=>

It was again a very good session! I did get them some vegetables to eat at the start and gave Rooz some rice in the middle and bread for Marc and grapes for Cyrus. I continue to feel they are getting a good feel for the basics of Linear Algebra. I am also getting a good feeling and feel proud that I can communicate these ideas to them.

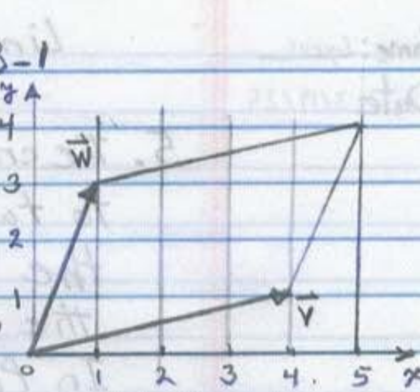


Name: Cyrus

Date: 3/19/25

Linear Algebra Problems 8-1

1. The $\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$



For matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

Find $\det A = 11$

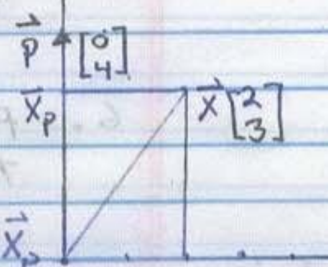
Is this the area of the Parallelogram made by $\vec{v}$ & $\vec{w}$?

Yes NO

2. The dot product of vectors $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$

find the dot product of

$\vec{P} \cdot \vec{X} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$



Find the Projection of $\vec{X}$ on $\vec{P}$

$\|\vec{X}_p\| =$ 3

if we multiply the length of $\vec{P}$ by $\vec{X}_p$

what do we get? $\|\vec{P}\| \|\vec{X}_p\| = 12$

is this equal to $\vec{P} \cdot \vec{X}$ Yes No

3. For $\vec{P} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ find $\vec{P} \cdot \vec{X} = x + 2y + 3z$

4. The determinant for a 3x3 matrix is calculated in 3 Steps

$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ $\det A \Rightarrow$ $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$

$1 \times \det\begin{bmatrix} 4 & 5 \\ 2 & 4 \end{bmatrix} - 1 \times \det\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix} + 1 \times \det\begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix}$

$\det A = 3$

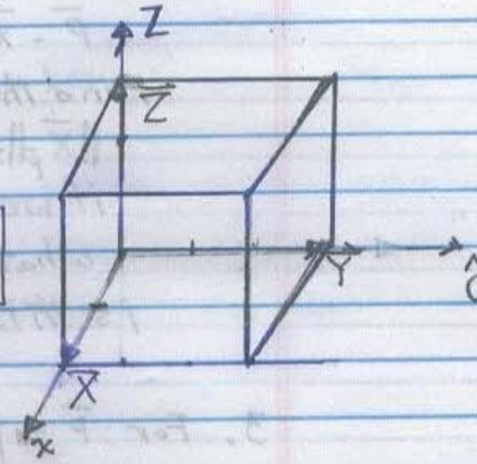
Name: Cyrus
Date: 3/19/25

Linear Algebra Problems 8-2

5. The cross product is a vector that is perpendicular to two vectors and its length is equal to the determinant of the matrix A made by the two vectors.
   In Problem 1 $\vec{V} \times \vec{W} = \vec{P}$.
   $\vec{P}$ is in the z-direction
   and the length of $\vec{P} = \underline{11}$

6. Find the volume of the rectangular Prism made by vector

$$\vec{X} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \vec{Y} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \quad \vec{Z} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$



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7. Find the volume of the parallelopiped made by

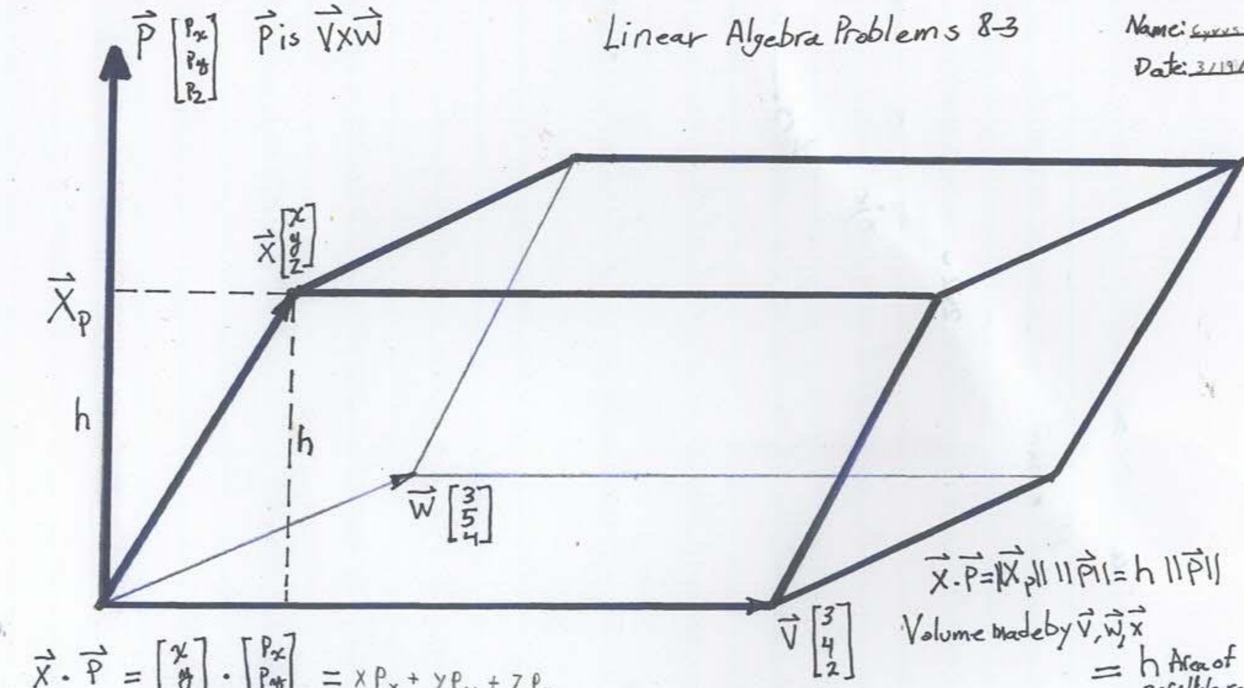
$$\vec{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{V} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \text{ and } \vec{W} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 2 & 5 \end{bmatrix} \qquad 3$$

Linear Algebra Problems 8-3

Name: Cyrus

Date: 3/19/25



$\vec{X}\cdot\vec{P} = \|\vec{X}_p\|\,\|\vec{P}\| = h\,\|\vec{P}\|$

Volume made by $\vec{v}, \vec{w}, \vec{x}$ $= h$ Area of parallelogram made by $\vec{V}$ & $\vec{W}$

$$\vec{x}\cdot\vec{P} = \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}P_x\\P_y\\P_z\end{bmatrix} = xP_x + yP_y + zP_z$$

Volume of the parallelopiped made by $\vec{v}, \vec{w}, \vec{x}$ $= \det\left(\begin{bmatrix}x&3&3\\y&4&5\\z&2&4\end{bmatrix}\right)$

$$= x\det\left(\begin{bmatrix}4&5\\2&4\end{bmatrix}\right) - y\det\left(\begin{bmatrix}3&3\\2&4\end{bmatrix}\right) + z\det\left(\begin{bmatrix}3&3\\4&5\end{bmatrix}\right)$$

$P_x = 6$

$P_y = -6$

$P_z = 3$

$$= 6x - 6y + 3z$$

$$\|\vec{P}\| = \sqrt{P_x^2 + P_y^2 + P_z^2} = \sqrt{36+36+9} = \sqrt{81} = 9$$

What you will review and/or learn today:

1. The meaning of determinant for a 2x2 and a 3x3 matrix.
2. How to compute the determinant for a 2x2 or a 3x3 matrix.
3. The meaning of the dot product as a projection.
4. How to compute the dot product of two vectors.
5. The meaning of the cross product of two vectors.
6. How to compute and find the cross product of 2 vectors in xy plane
7. How to find the volume of a rectangular Prism.
8. How to find the volume of a parallelopiped.
9. How to compute and find the cross product of 2 vectors in xyz space
10. How we use duality to find dot and cross products!

Name: RD02
Date: 3-19-25

Linear Algebra Problems 8-1

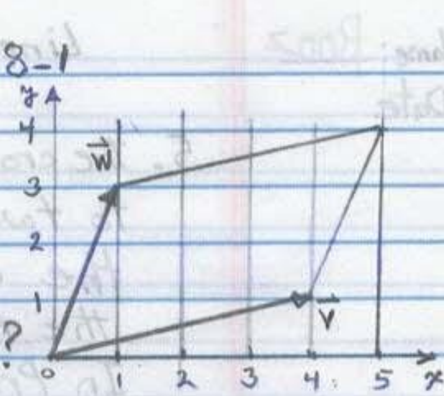
1. The $\det\begin{bmatrix}a&c\\b&d\end{bmatrix} = ad - bc$

For matrix $A = \begin{bmatrix}4&1\\1&3\end{bmatrix}$

Find $\det A = 11$

Is this the area of the Parallelogram made by $\vec{v}$ & $\vec{w}$?

Yes NO



2. The dot product of vectors $\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}c\\d\end{bmatrix} = ac + bd$

find the dot product of

$\vec{P}\cdot\vec{X} = \begin{bmatrix}0\\4\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}12\\0\end{bmatrix}$

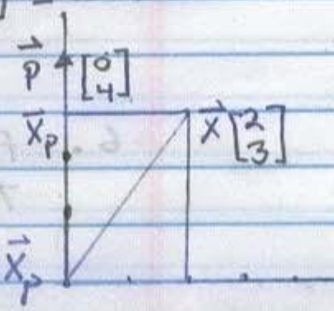
Find the Projection of $\vec{X}$ on $\vec{p}$

$\|\vec{X}_p\| = \underline{3}$

if we multiply the length of $\vec{P}$ by $\vec{X}_p$

what do we get? $\|\vec{P}\|\|\vec{x}_p\| = 12$

is this equal to $\vec{P}\cdot\vec{X}$ Yes No



3. For $\vec{P} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\vec{X} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ find $\vec{P}\cdot\vec{X} = x + 2y + 3z$

4. the determinat for a 3x3 matrix is calculated in 3 Steps

$A = \begin{bmatrix}1&3&3\\1&4&5\\1&2&4\end{bmatrix}$ $\det A \Rightarrow$ $\begin{bmatrix}1&3&3\\1&4&5\\1&2&4\end{bmatrix}$ $\begin{bmatrix}1&3&3\\1&4&5\\1&2&4\end{bmatrix}$ $\begin{bmatrix}1&3&3\\1&4&5\\1&2&4\end{bmatrix}$

$1\times\det\begin{bmatrix}4&5\\2&4\end{bmatrix} - 1\times\det\begin{bmatrix}3&3\\2&4\end{bmatrix} + 1\times\det\begin{bmatrix}3&3\\4&5\end{bmatrix}$

$\det A = 1\times 6 - 6 + 3 = 3$

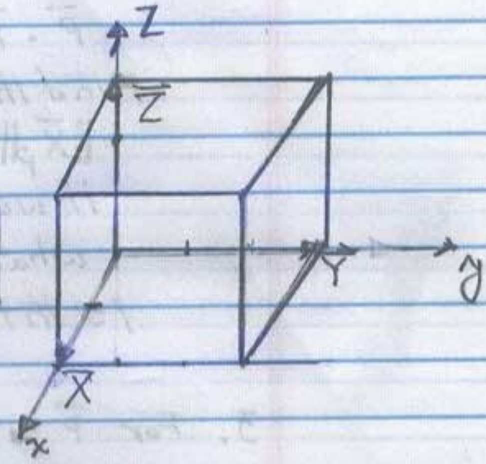
3

Name: Rooz
Date:

Linear Algebra Problems 8-2

5. The cross product is a vector that is perpendicular to two vectors and its length is equal to the determinant of the matrix A made by the two vectors.
   In Problem 1 $\vec{V} \times \vec{W} = \vec{P}$
   $\vec{P}$ is in the z-direction
   and the length of $\vec{P}$ = 11

6. Find the volume of the rectangular Prism made by vector
   $\vec{X} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\vec{Y} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ $\vec{Z} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$



   18

7. Find the volume of the parallelopiped made by
   $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{V} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ and $\vec{W} = \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}$

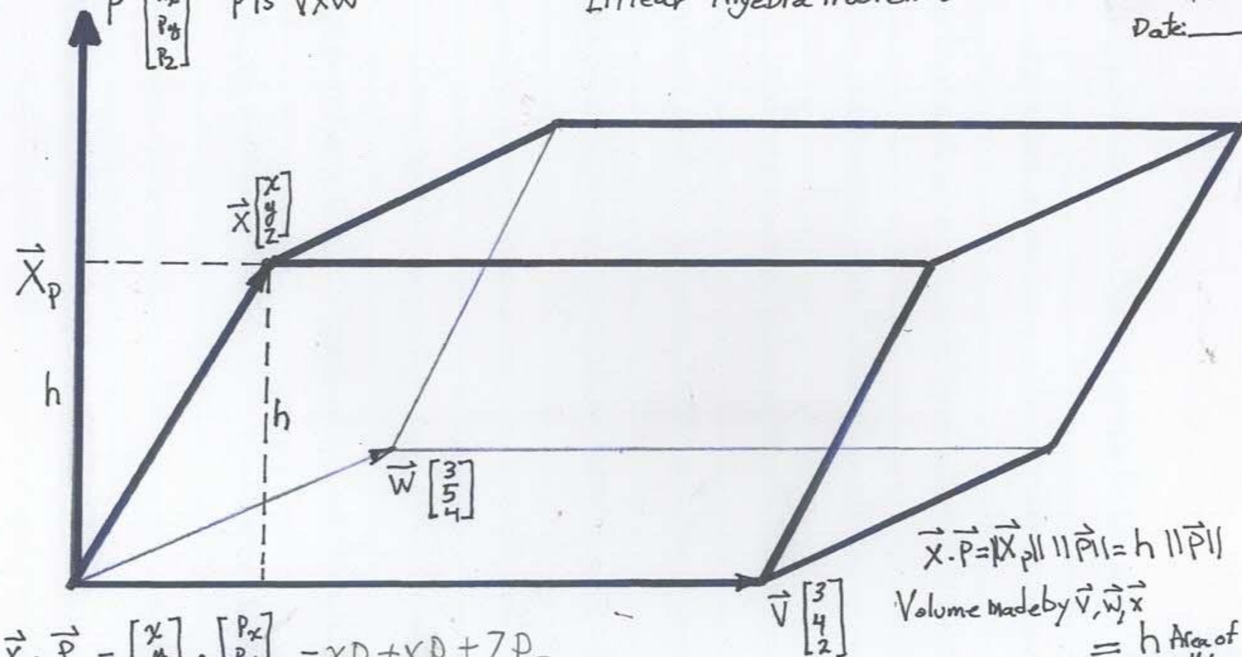
   $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$

Linear Algebra Problems 8-3

Name: Rodz

Date: ______

$\vec{P} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$ $\vec{P}$ is $\vec{V} \times \vec{W}$



$\vec{X} \cdot \vec{P} = \|\vec{X}_p\| \, \|\vec{P}\| = h \, \|\vec{P}\|$

Volume made by $\vec{V}, \vec{W}, \vec{X}$

$= h$ Area of parallelogram made by $\vec{V}$ & $\vec{W}$

$$\vec{x} \cdot \vec{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = xP_x + yP_y + zP_z$$

Volume of the parallelopiped made by $\vec{v}, \vec{w}, \vec{x}$ $= \det\left(\begin{bmatrix} x & 3 & 3 \\ y & 4 & 5 \\ z & 2 & 4 \end{bmatrix}\right)$

$$= x\det\left(\begin{bmatrix} 4 & 5 \\ 2 & 4 \end{bmatrix}\right) - y\det\left(\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix}\right) + z\det\left(\begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix}\right)$$

$P_x = 6$

$P_y = -6$

$P_z = 3$

$$= 6x - 6y + 3z$$

$$\|\vec{P}\| = \sqrt{P_x^2 + P_y^2 + P_z^2} =$$