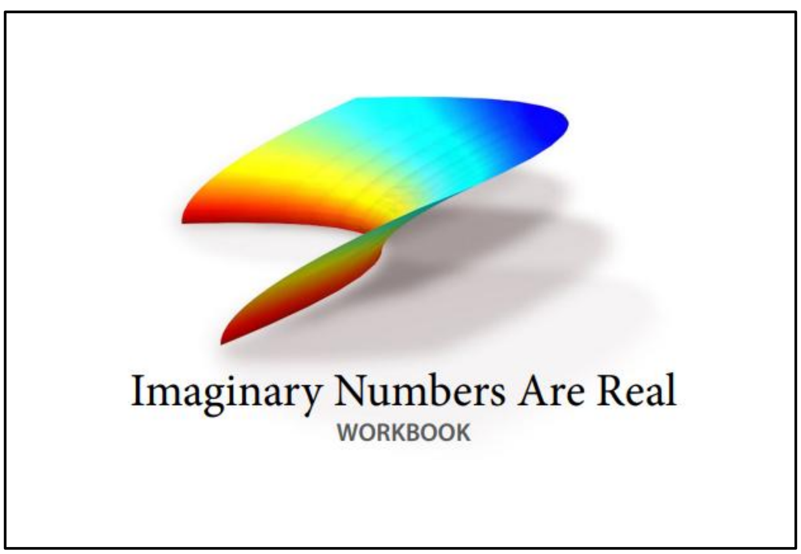
Math Circle Adventures with *Imaginary Numbers are Real*

Math Circle Notes

Eight Wednesday afternoons • August–October 2025

Three young mathematicians proudly displayed the Riemann Surfaces they built during our Math Circle.



Inspired by Stephen Welch's *Imaginary Numbers are Real*

Table of Contents

Introduction to Our Adventures with *Imaginary Numbers are Real*

In August and September of 2025, three young mathematicians—Cyrus, Marc, and Rooz—began an extraordinary mathematical journey.

Each Wednesday after school we met for a little over an hour to explore Stephen Welch's remarkable video series Imaginary Numbers are Real. Over the course of eight weekly Math Circle Adventures, these students—two eleven-year-olds and a nine-year-old—were introduced to ideas that many students do not encounter until college. They learned about the history of numbers, the invention of imaginary numbers, Cardano's solution of the cubic equation, Euler's Formula, Riemann surfaces, and finally Chaos Theory and the Mandelbrot Set.

The primary purpose of these notes is simply to preserve the story of those Wednesday afternoon Adventures.

Almost every page grew directly from the notes I wrote immediately after each Math Circle session. Rather than rewriting those experiences into a traditional textbook, I have simply organized and edited them so that they tell the story of what actually happened. The conversations are real. The questions are real. The discoveries are real. The completed worksheets and photographs included throughout the book are part of that story.

These notes are not intended to replace Stephen Welch's Imaginary Numbers are Real videos or his excellent companion workbook. In my opinion, the videos together with the companion PDF provide everything an educator needs to introduce this remarkable subject to motivated students. The companion workbook contains thoughtful discussions, carefully designed exercises, and challenging problems that beautifully complement the videos. I strongly recommend that every teacher, parent, homeschool family, or Math Circle leader use Stephen's original materials as the primary resource.

Almost allof the worksheet activities completed during our Math Circle were selected from, or inspired by, Stephen's companion workbook. Rather than reproducing those materials here, I have chosen to include only our completed student work, photographs, and observations from the sessions themselves. My hope is that these notes demonstrate one way in which Stephen's remarkable work can be successfully adapted for gifted elementary and middle-school students.

Most of all, I hope these pages preserve the memory of eight wonderful Wednesday afternoons spent exploring beautiful mathematics together. Years from now, I hope Cyrus, Marc, and Rooz will look back at these Adventures, remember the excitement of discovering ideas that once seemed impossible, and perhaps smile as they recall the Riemann surfaces they proudly carried home.

Finally, I would like to express my sincere appreciation to Stephen Welch. His extraordinary ability to communicate difficult mathematics with clarity, history, and beautiful visualizations inspired these Adventures and made them possible. I hope these notes encourage many more students, parents, teachers, and Math Circle leaders to discover his remarkable work.

Adventure 1 — A New Way to Find the Roots of a Quadratic

Date: Wednesday, August 6, 2025

Time: 1:50 PM – 2:50 PM

Participants: Cyrus, Rooz, and Dar

Absent: Marc

Introduction

This Adventure serves as the introduction to our journey through Stephen Welch's remarkable **Imaginary Numbers are Real** series. Before beginning that journey, however, I wanted the students to see a different and highly visual way of thinking about quadratic equations.

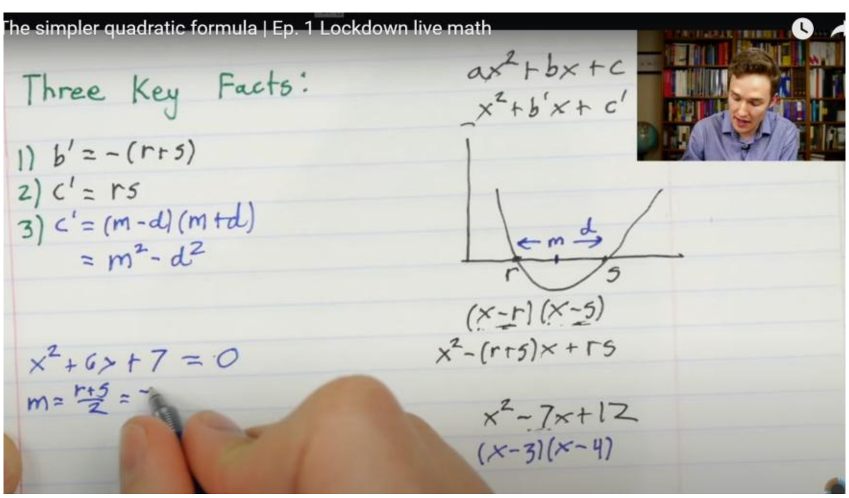
Rather than relying immediately on the familiar quadratic formula, we explored an elegant method for finding the roots of a quadratic by using the **mean of the roots** and their **distance from the mean**. This approach helped the students visualize what the roots represent and prepared them for the much deeper question that would arise the following week:

What happens when a quadratic equation has no real roots?

Although this Adventure is not yet about imaginary numbers, it lays the foundation for everything that follows.

The Adventure

We began the afternoon by watching approximately thirty minutes of the **first live session by 3Blue1Brown**, which presents an elegant alternative method for finding the roots of a quadratic equation. The students immediately saw that there is often more than one way to approach a mathematical problem and that understanding the geometry behind a formula can be just as important as learning the formula itself.



I hope Marc will have an opportunity to watch this video before our next meeting so that he can complete the same activities as the rest of the group.

After the video, we worked through three of the problems that I had prepared. Cyrus and Rooz each completed the activities, and I helped them along the way, particularly Rooz whenever he needed additional guidance. Their completed work is included later in this Adventure.

Overall, it was a very enjoyable session, although we certainly missed having Marc with us. Since he was absent, I intentionally introduced a few "mistakes" into the examples to see whether anyone would notice them. Cyrus quickly found several of them, much to my amusement. It made for a lively discussion and reminded all of us that careful observation is one of the most important habits in mathematics.

By the end of the afternoon, the students had discovered a new way of thinking about quadratic equations—one that emphasized understanding rather than memorization.

Next week we will begin working through Stephen Welch's wonderful series **Imaginary Numbers are Real**, where a simple quadratic equation,

$$x^2 + 1 = 0,$$

will lead us into one of the most beautiful stories in mathematics.

What I Learned

his was an excellent way to begin our journey. The students saw that there are often several ways to solve the same mathematical problem and that understanding is more important than memorizing a formula. Although we missed Marc, Cyrus and Rooz remained engaged throughout the afternoon, and Cyrus enjoyed catching several of the intentional mistakes that I had included in the examples. Next week we begin Stephen Welch's **Imaginary Numbers are Real** series.

Looking Ahead

Next Wednesday we begin Stephen Welch's **Imaginary Numbers are Real** series.

For the first time the students will encounter a quadratic equation that has **no real solutions**. Rather than viewing this as the end of the story, they will discover that it marks the beginning of an entirely new chapter in the history of mathematics.

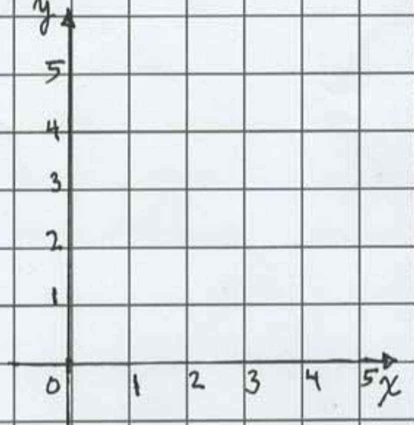
1cm METRIC GRAPH PAPER

4692

$y = x^2 - 2mx + p$

Graph $y = x^2 - 4x + 3$

x	y
0	
1	
2	
3	
4	



if x_1 & x_2 are the roots:

1. find m the mean of x_1 & x_2

$m = \frac{x_1 + x_2}{2} =$

2. find d where the roots are $x_1 = m + d$ $x_2 = m - d$

$d = ?$

Use $(x-d)(x+d) = p$

$d =$

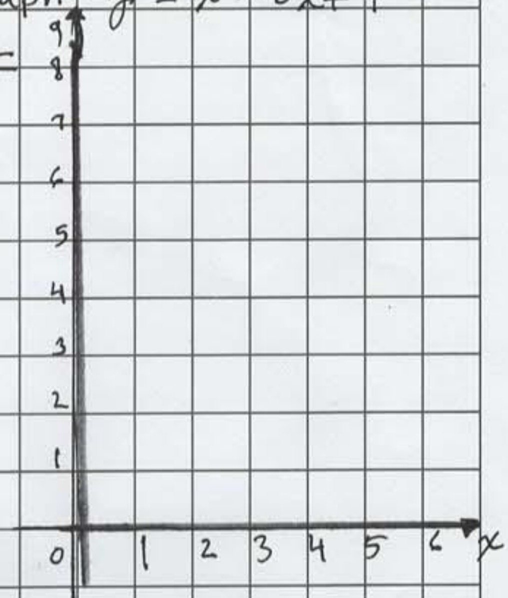
3. find x_1 & x_2

$x_1 =$

$x_2 =$

Graph $y = x^2 - 6x + 9$

x	y
0	
1	
2	
3	
4	
5	

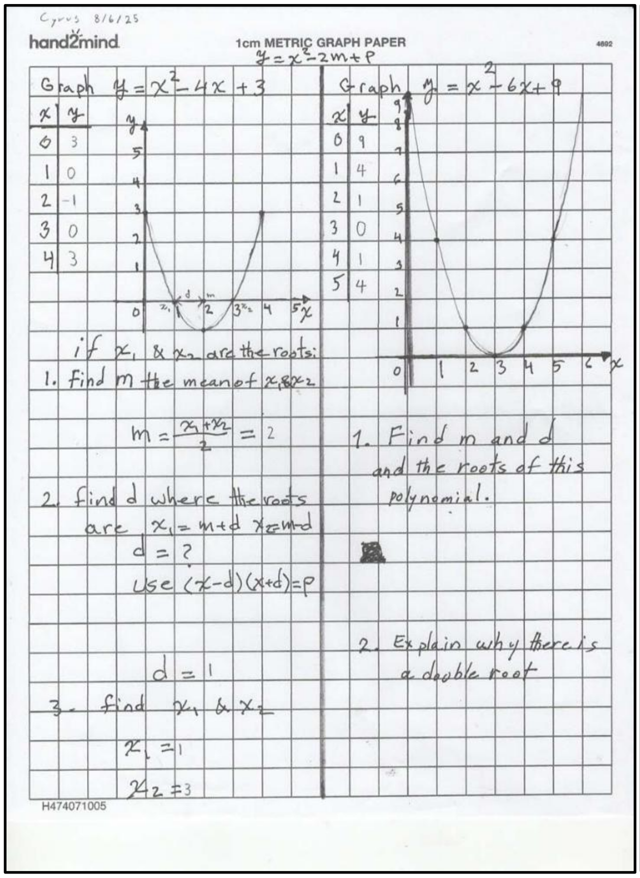


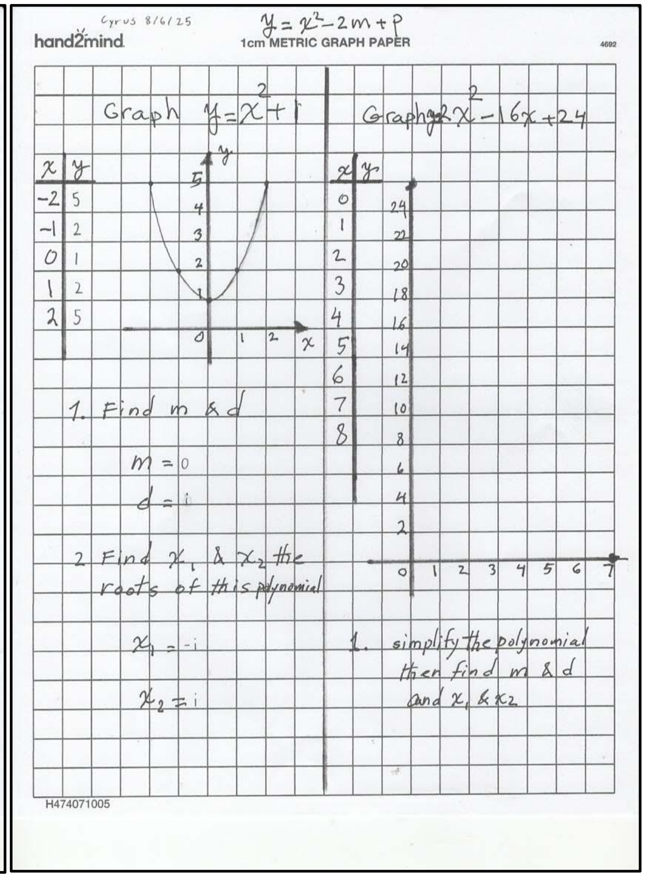
1. Find m and d and the roots of this polynomial.

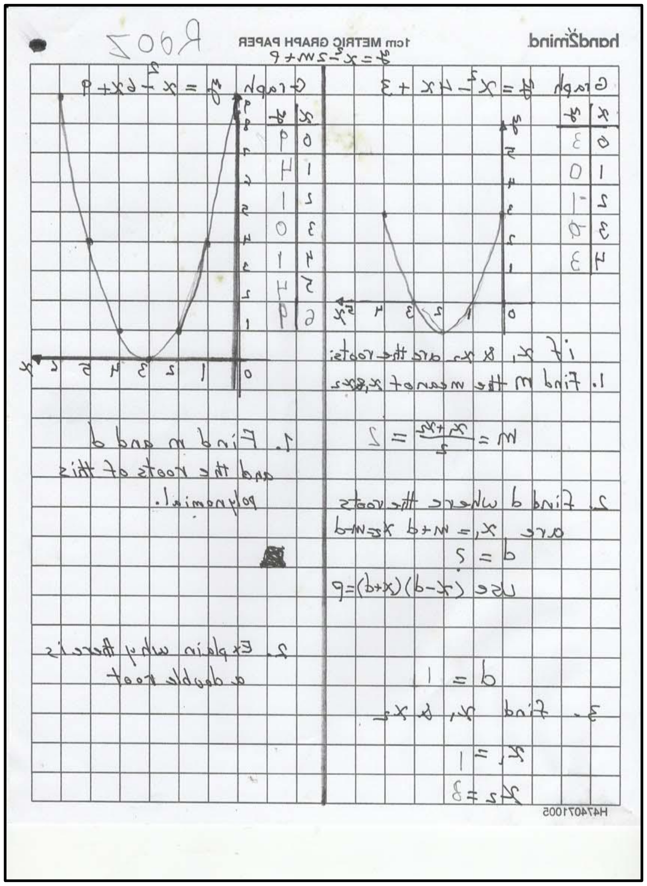
2. Explain why there is a double root

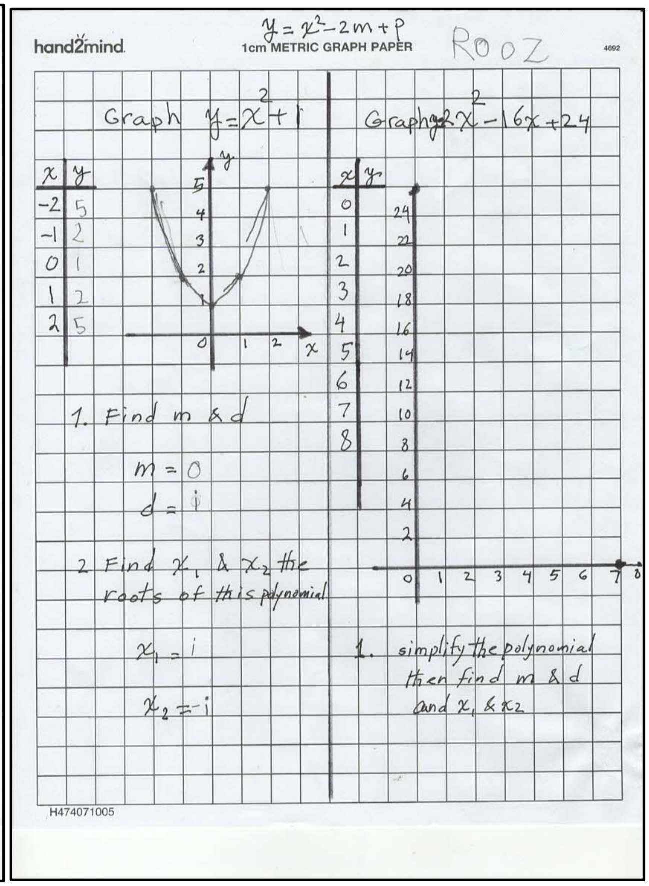
H474071005

Completed Student Worksheet









Adventure 2 — The Impossible Equation

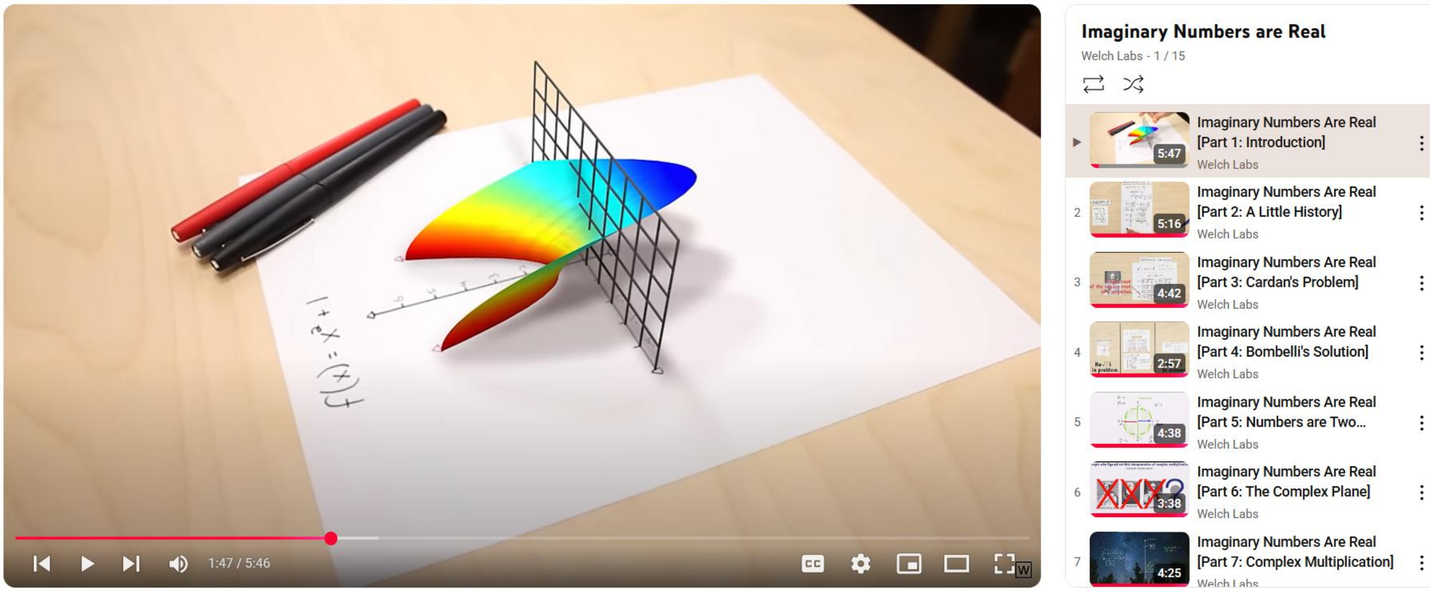
Date: Wednesday, August 13, 2025

Time: 1:55 PM – 3:10 PM

Participants: Cyrus, Marc, Rooz, and **Dr. Super**

Introduction

This week we began Stephen Welch's **Imaginary Numbers are Real** series. The first seven videos introduce the history of numbers and explain how mathematicians were eventually led to the invention of imaginary numbers while trying to solve cubic equations. Rather than rushing through the material, we frequently stopped the videos to discuss the mathematics, the history, and many of the questions suggested in Stephen's accompanying workbook.



The Adventure

We watched **Videos 1–7** from the *Imaginary Numbers are Real* series. Each video is only about five minutes long and begins with a brief review of the previous one, making it easy to pause and discuss the ideas before continuing. I have now watched these videos several times myself, and I believe the students will benefit from seeing them again as the series progresses.

The students especially enjoyed the historical development of numbers and the story of how imaginary numbers first appeared while mathematicians were trying to solve third-degree equations. We frequently stopped the videos to discuss important ideas, and I also used several of the discussion questions suggested in Stephen Welch's workbook. I purchased a copy of the workbook and found it to be an excellent companion to the videos.

After finishing the videos, the students completed two pages of exercises from the first section of the workbook. These problems reviewed graphing quadratic functions and finding their roots—excellent practice that will also help them in Algebra. We decided to postpone the more challenging exercises involving complex numbers until our next Adventure.

This was one of our longer Math Circle sessions. While we watched the videos, I served plenty of vegetables and fresh baguette bread to keep everyone's energy up. The strategy worked well—the students remained focused throughout the entire session. As usual, I helped Rooz whenever he needed a little assistance, but he kept up with the group and did a very good job.

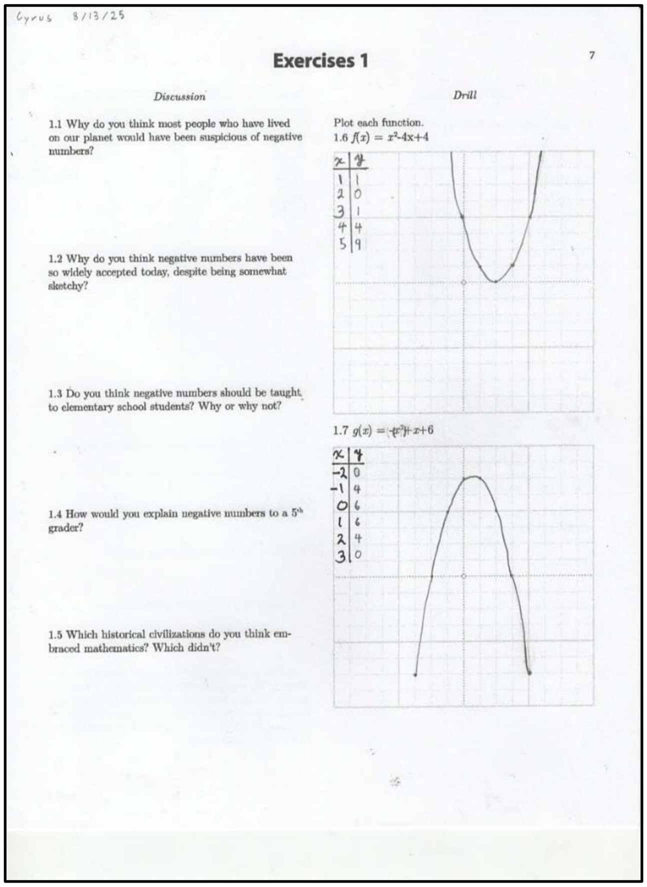
What I Learned

The first seven videos provide an excellent historical introduction to the subject and generated many good discussions. The combination of short videos, frequent pauses, and workbook exercises worked well. Rather than trying to cover too much material in one afternoon, it was better to let the students absorb the history and become comfortable with the ideas before moving on to complex numbers.

Looking Ahead

Next week we will continue the series and begin working more directly with complex numbers. The students will complete additional workbook problems and take their first real steps into the fascinating world of imaginary numbers.

Completed Activity Sheets



Cyrus 8/13/25

Exercises 1

7

Discussion

1.1 Why do you think most people who have lived on our planet would have been suspicious of negative numbers?

1.2 Why do you think negative numbers have been so widely accepted today, despite being somewhat sketchy?

1.3 Do you think negative numbers should be taught to elementary school students? Why or why not?

1.4 How would you explain negative numbers to a 5th grader?

1.5 Which historical civilizations do you think embraced mathematics? Which didn't?

Drill

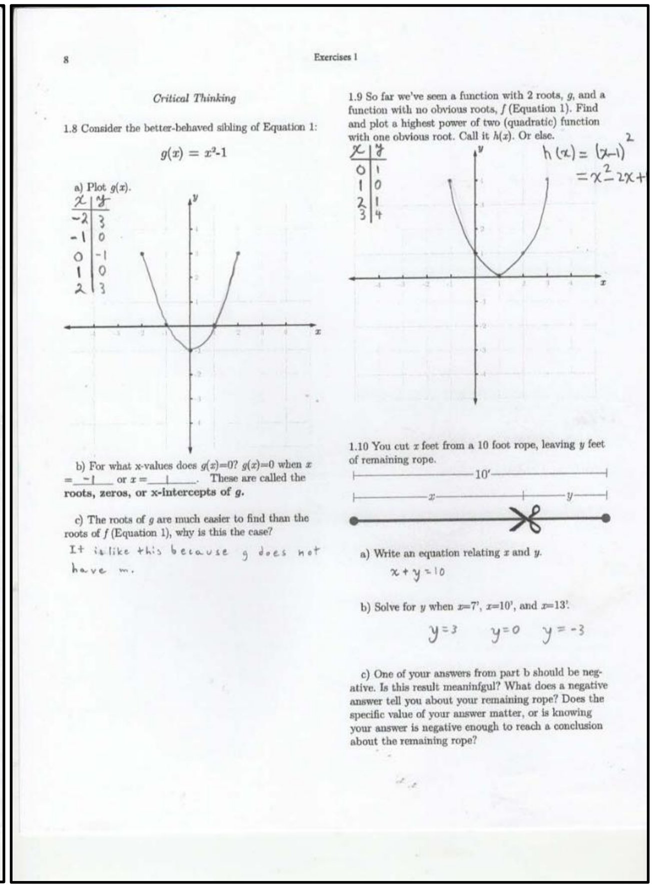
Plot each function.

1.6 $f(x) = x^2-4x+4$

x	y
1	1
2	0
3	1
4	4
5	9

1.7 $g(x) = -x^2+x+6$

x	y
-2	0
-1	4
0	6
1	6
2	4
3	0



8 Exercises 1

Critical Thinking

1.8 Consider the better-behaved sibling of Equation 1:

$$g(x) = x^2-1$$

a) Plot $g(x)$.

x	y
-2	3
-1	0
0	-1
1	0
2	3

b) For what x-values does $g(x)=0$? $g(x)=0$ when x = __-1__ or x = __1__. These are called the **roots, zeros, or x-intercepts of g.**

c) The roots of g are much easier to find than the roots of f (Equation 1), why is this the case?

It is like this because g does not have m.

1.9 So far we've seen a function with 2 roots, g, and a function with no obvious roots, f (Equation 1). Find and plot a highest power of two (quadratic) function with one obvious root. Call it $h(x)$. Or else.

x	y
0	1
1	0
2	1
3	4

$h(x) = (x-1)^2$
$= x^2-2x+$

1.10 You cut x feet from a 10 foot rope, leaving y feet of remaining rope.

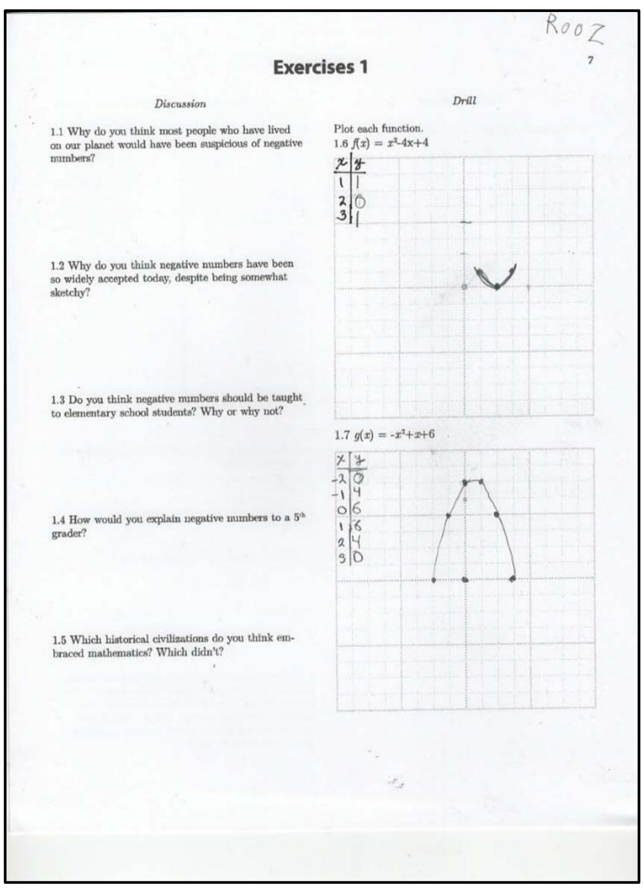
a) Write an equation relating x and y.

$x+y=10$

b) Solve for y when x=7', x=10', and x=13'.

$y=3$ $y=0$ $y=-3$

c) One of your answers from part b should be negative. Is this result meaninfgul? What does a negative answer tell you about your remaining rope? Does the specific value of your answer matter, or is knowing your answer is negative enough to reach a conclusion about the remaining rope?



Rooz

Exercises 1

7

Discussion

1.1 Why do you think most people who have lived on our planet would have been suspicious of negative numbers?

1.2 Why do you think negative numbers have been so widely accepted today, despite being somewhat sketchy?

1.3 Do you think negative numbers should be taught to elementary school students? Why or why not?

1.4 How would you explain negative numbers to a 5th grader?

1.5 Which historical civilizations do you think embraced mathematics? Which didn't?

Drill

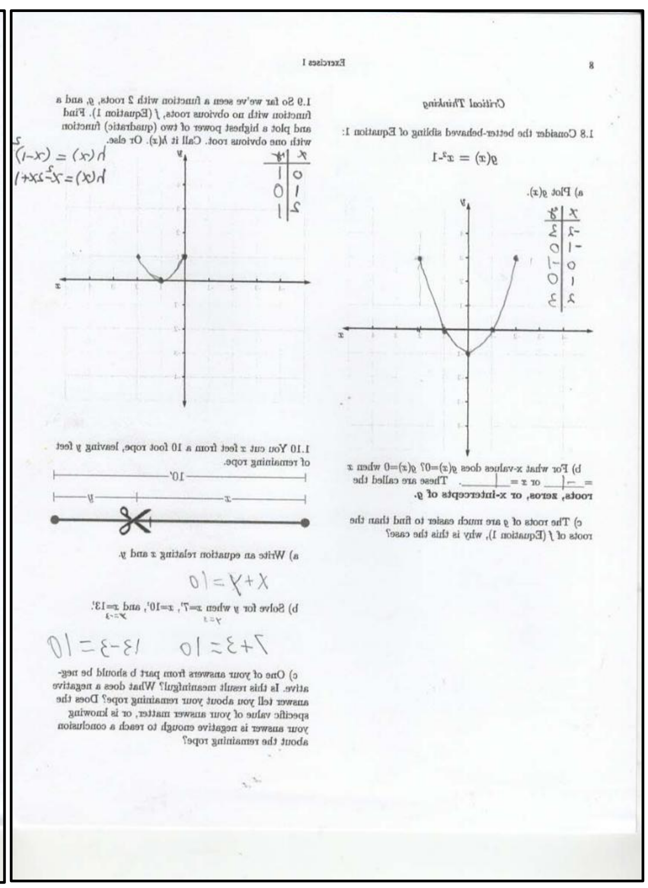
Plot each function.

1.6 $f(x) = x^2-4x+4$

x	y
1	1
2	0
3	1

1.7 $g(x) = -x^2+x+6$

x	y
-2	0
-1	4
0	6
1	6
2	4
3	0



8 Exercises 1

Critical Thinking

1.8 Consider the better-behaved sibling of Equation 1:

$$g(x) = x^2-1$$

a) Plot $g(x)$.

x	y
-2	3
-1	0
0	-1
1	0
2	3

b) For what x-values does $g(x)=0$? $g(x)=0$ when x = ____ or x = ____. These are called the **roots, zeros, or x-intercepts of g.**

c) The roots of g are much easier to find than the roots of f (Equation 1), why is this the case?

1.9 So far we've seen a function with 2 roots, g, and a function with no obvious roots, f (Equation 1). Find and plot a highest power of two (quadratic) function with one obvious root. Call it $h(x)$. Or else.

x	y
0	1
1	0
2	1

$h(x) = (x-1)^2$
$h(x) = x^2-2x+1$

1.10 You cut x feet from a 10 foot rope, leaving y feet of remaining rope.

a) Write an equation relating x and y.

$X+Y=10$

b) Solve for y when x=7', x=10', and x=13'.

$7+3=10$ $13-3=10$

c) One of your answers from part b should be negative. Is this result meaninfgul? What does a negative answer tell you about your remaining rope? Does the specific value of your answer matter, or is knowing your answer is negative enough to reach a conclusion about the remaining rope?

Adventure 3 — Cardano's Quest

Date: Wednesday, August 20, 2025

Time: 1:50 PM – 3:05 PM

Participants: Cyrus, Marc, Rooz, and Dr. Super

Introduction

This Adventure became one of the most exciting of the entire series. Before our meeting, I spent much of the week trying to understand how **Cardano's Formula** was actually discovered. Like many students, I had learned the formula years ago without ever seeing where it came from. I wanted the students to experience not only the mathematics but also the excitement of uncovering one of the great mathematical discoveries in history.

The Adventure

During the week I watched several additional videos, including **How Imaginary Numbers Were Invented**, which beautifully explains how second- and third-degree equations were solved graphically and how these ideas eventually led to Cardano's Formula. Cyrus had already watched part of this video the previous week, and I recommended it to the rest of the group.

As I picked the students up after school, they could immediately see how excited I was. I told them that ever since I was in the eleventh or twelfth grade, I had wanted to understand how this remarkable formula had been discovered. By this point I felt that I had solved about ninety percent of the puzzle, and I couldn't wait to share it with them. Looking back, this was probably the most exciting and challenging session for me, and I think the students appreciated both the mathematics and my enthusiasm. Rooz found some of the ideas more difficult than Cyrus and Marc, so I planned to review them with him again the following week. Since we had already worked for well over an hour, I decided to skip the workbook problems I had prepared and concentrate on understanding the main ideas.

We next explored several examples using **Algebra Lab Gear**, which gave the students a visual way of understanding the ideas behind Cardano's Formula. I then gave them two pattern puzzles. They solved them surprisingly quickly once they recognized the underlying patterns. Rooz also did very well on these activities. Cyrus asked an excellent question about second-degree equations with solutions other than one, and I promised to return to that question the following week. We then looked at Cardano's Formula itself and found where the same numbers from the puzzles appeared in the formula for two different cubic equations. I told the students that next week I would show them the complete derivation for this special class of cubic equations.

Before returning to the videos, we reviewed several trigonometric ideas that Cyrus and Marc had studied previously and that Rooz had begun learning the week before. This review would prepare them for the upcoming videos on multiplying complex numbers. We then watched Videos 6 and 7 once more before continuing with Videos 8 and 9. I felt the review was important because these ideas need time to develop and should not be rushed.

Throughout the videos we paused frequently to discuss the addition and multiplication of complex numbers, especially for Rooz, who was seeing many of these ideas for the first time. We also spent time discussing the concept of **closure** and how different number systems are closed under different operations. I thought this provided an excellent conclusion to the first part of Stephen Welch's series. The final four videos are considerably more sophisticated, and I left the session wondering how best to present them. Even after watching them several times myself, I was still discovering new ideas. Nevertheless, I felt that simply exposing the students to these beautiful concepts might inspire them to continue exploring higher mathematics in the future.

What I Learned

This Adventure reminded me that enthusiasm is contagious. The students could tell how excited I was about finally understanding Cardano's Formula, and that excitement seemed to carry over into our discussions. Rather than trying to cover every exercise, it was more valuable to slow down, explore the ideas visually, and allow the students to appreciate the beauty behind the mathematics.

Looking Ahead

Next week we will derive Cardano's Formula for this special class of cubic equations and continue our study of complex numbers. We will also return to several of the workbook problems that we postponed during this Adventure.

Figure 3.1 – Imaginary Numbers are Real Videos.

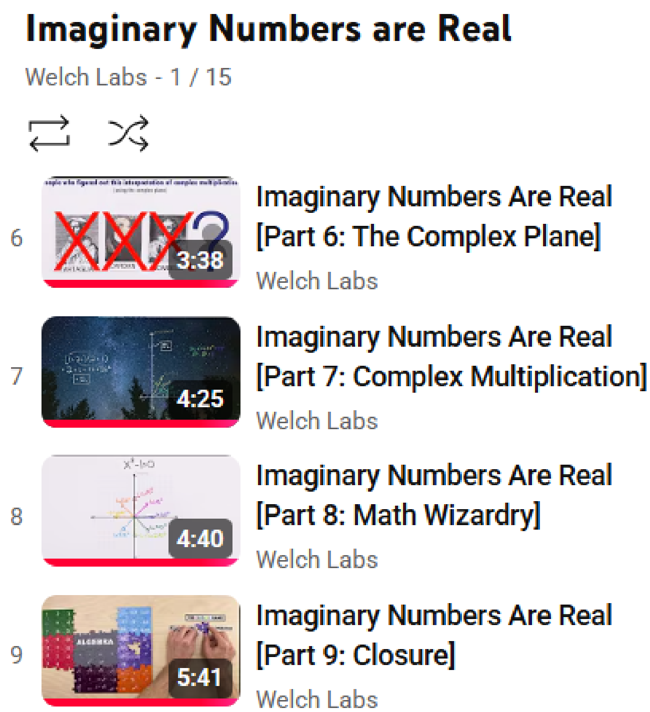


Figure 3.2 – Algebra Lab Gear demonstration of the cubic equation.

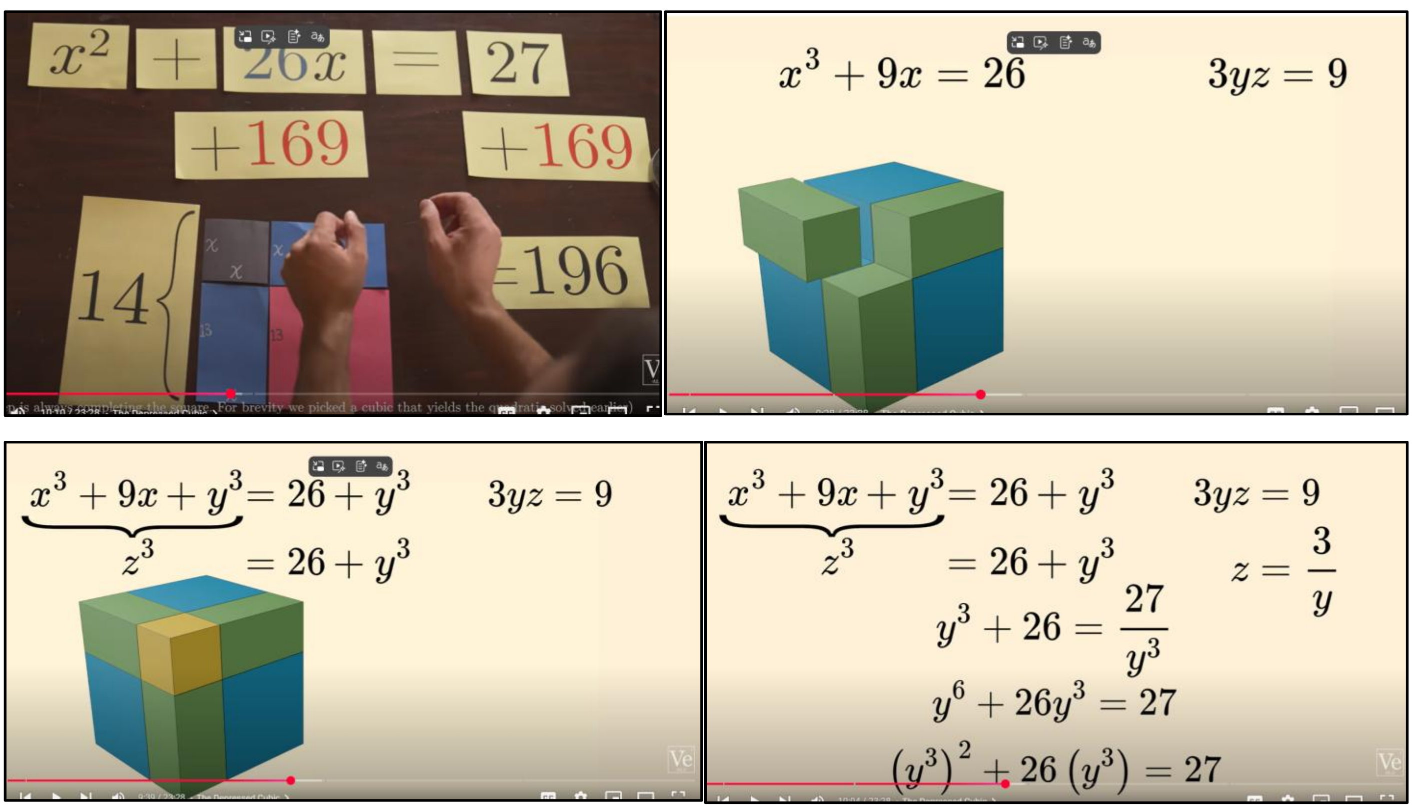


Figure 3.3 – Pattern puzzle on cubic equations with Solution.

Find the Patterns and Fill in the Missing Coefficients
for each of these Third Degree Equations.
Make Sure that the Value that you Found for X
Works for that Equation.

X^3	+	$6X$	=	7	X=1
X^3	+	X	=	26	X=
X^3	+	$12X$	=		X=
X^3	+	$15X$	=	124	X=4
X^3	+	X	=		X=
X^3	+	X	=	342	X=
X^3	+	$24X$	=		X=
X^3	+	X	=	920	X=

$X^3 + 6X = 7$	$X=1$
$X^3 + 9X = 26$	$X=2$
$X^3 + 12X = 63$	$X=3$
$X^3 + 15X = 124$	$X=4$
$X^3 + 18X = 215$	$X=5$
$X^3 + 21X = 342$	$X=6$
$X^3 + 24X = 511$	$X=7$
$X^3 + 27X = 920$	$X=8$

Find and fill in the values for a and b in each row such that both a+b and a-b are cubes.

a	b	a+b	a-b
13	14	27	-1
31.5			
	63		
		216	-1
171			
	256.5		
364		729	

Figure 3.4 – Cardano's Formula examples.

$$x^3 = cx + d$$

Cardano's Formula:

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$$

$$x^3 = -9x + 26$$

$$x = \sqrt[3]{13 + \sqrt{169+27}} + \sqrt[3]{13 - \sqrt{196}}$$

$$= \sqrt[3]{13+14} + \sqrt[3]{13-14}$$

$$= \sqrt[3]{27} + \sqrt[3]{-1}$$

$$= 3 - 1 = 2$$

$$x^3 = -15x + 124$$

$$x = \sqrt[3]{62 + \sqrt{62^2 + 125}} + \sqrt[3]{62 - \sqrt{62^2 + 125}}$$

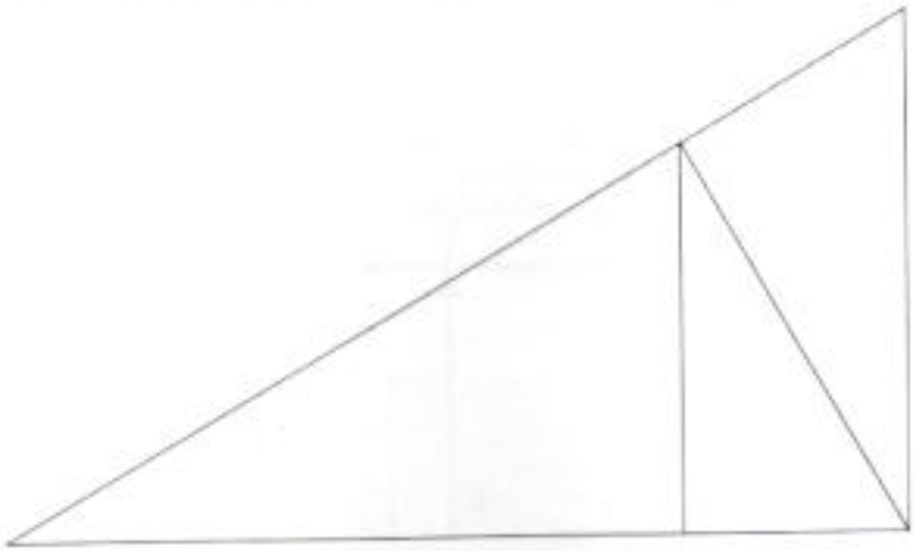
$$= \sqrt[3]{62 + \sqrt{3969}} + \sqrt[3]{62 - \sqrt{3969}}$$

$$= \sqrt[3]{62+63} + \sqrt[3]{62-63}$$

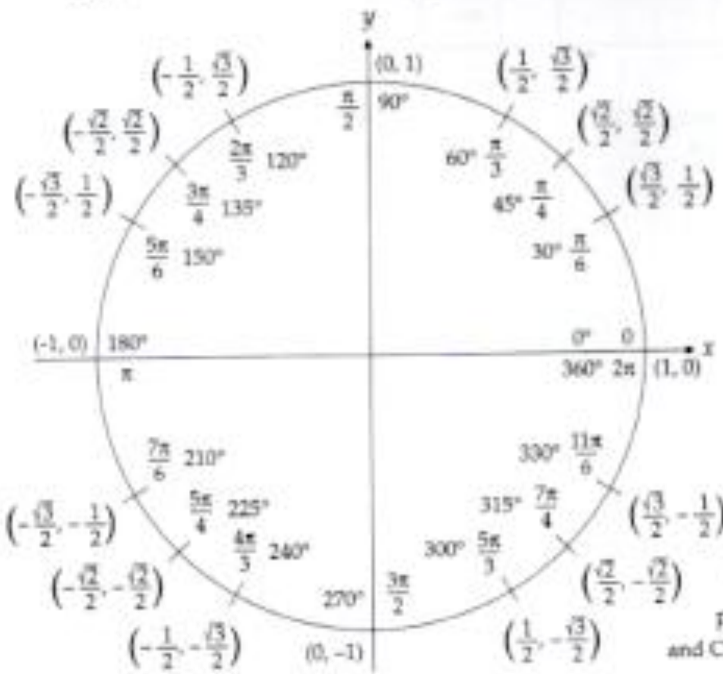
$$= \sqrt[3]{125} + \sqrt[3]{-1} = 5 - 1 = 4$$

Figure 3.5 – Trigonometric review.

Appendix 2. **Super Right Triangle**

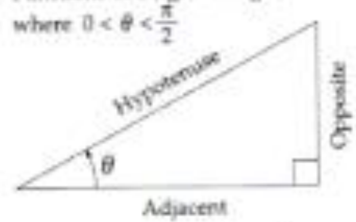


Appendix 3. **Trigonometric Functions**



◄ The unit circle, with important points expressed in degrees, radians, and Cartesian coordinates.

Functions of a right triangle, where $0 < \theta < \frac{\pi}{2}$



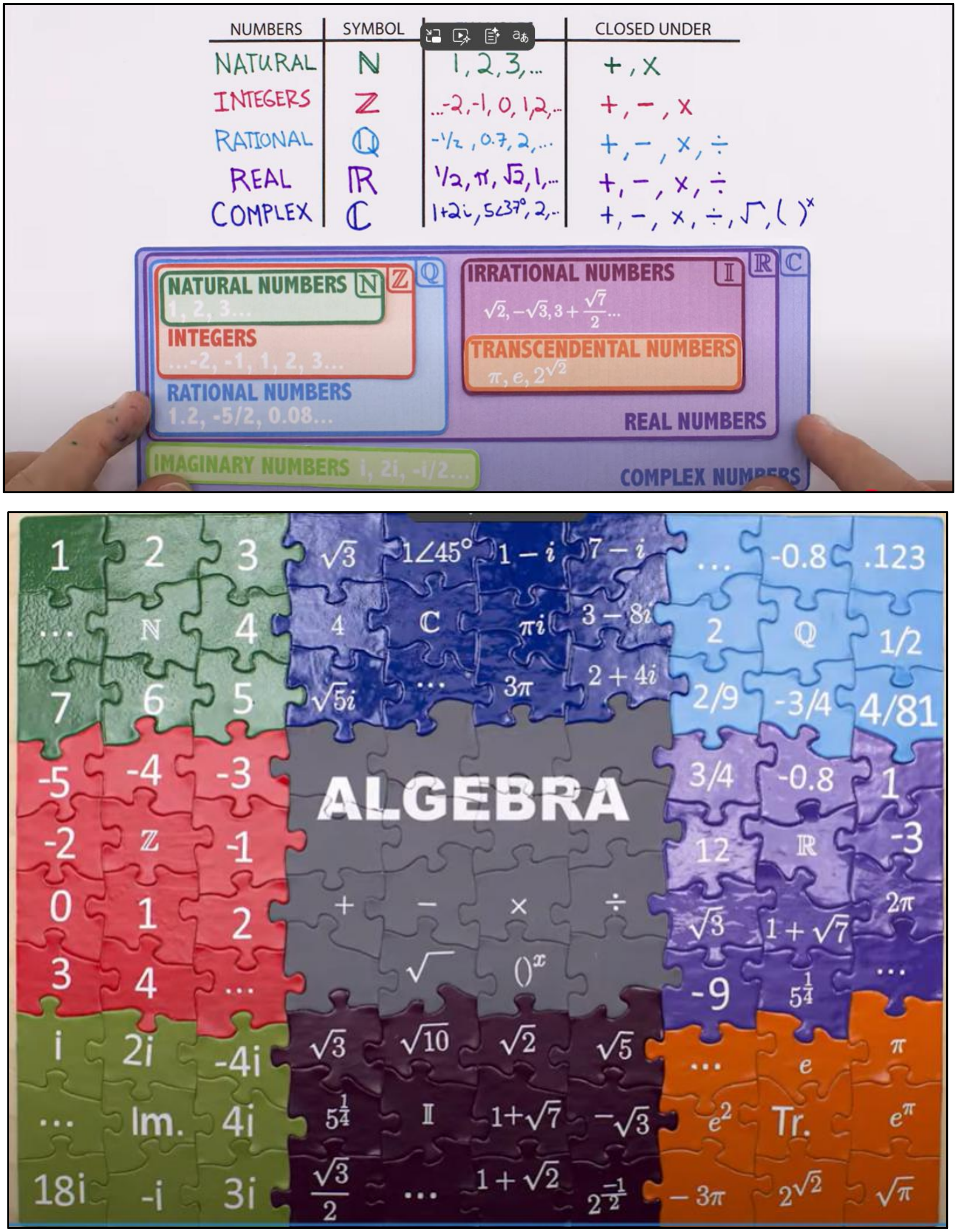
$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$ $\csc\theta = \frac{\text{hyp.}}{\text{opp.}}$

$\cos\theta = \frac{\text{adj.}}{\text{hyp.}}$ $\sec\theta = \frac{\text{hyp.}}{\text{adj.}}$

$\tan\theta = \frac{\text{opp.}}{\text{adj.}}$ $\cot\theta = \frac{\text{adj.}}{\text{opp.}}$

42 Exploring Trigonometry with Dr. Super's TRIGRAMS

Figure 3.6 – Number systems and closure.



Student Activities and Completed Student Work

Cyrus 8/20/25

Find the Patterns and Fill in the Missing Coefficients for each of these Third Degree Equations. Make Sure that the Value that you Found for X Works for that Equation.

$X^3 + 6X = 7$	$X=1$
$X^3 + 9X = 26$	$X=2$
$X^3 + 12X = 63$	$X=3$
$X^3 + 15X = 124$	$X=4$
$X^3 + 18X = 215$	$X=5$
$X^3 + 21X = 342$	$X=6$
$X^3 + 24X = 511$	$X=7$
$X^3 + 27X = 920$	$X=8$

Find and fill in the values for a and b in each row such that both a+b and a-b are cubes.

a	b	a+b	a-b
13	14	27	-1
31.5	32.5	64	-1
62	63	125	-1
107.5	108.5	216	-1
171	172	343	-1
255.5	256.5	512	-1
364	365	729	-1

Cyrus 8/20/25

$$x^3 = cx + d$$

Cardan's Formula:

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$$

$$x^3 = -9x + 26$$

$$x = \sqrt[3]{13 + \sqrt{169 + 27}} + \sqrt[3]{13 - \sqrt{196}}$$

$$= \sqrt[3]{13 + 14} + \sqrt[3]{13 - 14}$$

$$= \sqrt[3]{27} + \sqrt[3]{-1}$$

$$= 3 - 1 = 2$$

$$x^3 = -15x + 124$$

$$x = \sqrt[3]{62 + \sqrt{62^2 + 125}} + \sqrt[3]{62 - \sqrt{62^2 + 125}}$$

$$= \sqrt[3]{62 + \sqrt{3969}} + \sqrt[3]{62 - \sqrt{3969}}$$

$$= \sqrt[3]{62 + 63} + \sqrt[3]{62 - 63}$$

$$= \sqrt[3]{125} + \sqrt[3]{-1} = 5 - 1 = 4$$

Adventure 4 —Second and Third Degree Equations: Cardano's Method

Date: Wednesday, August 27, 2025

Time: 1:55 PM – 3:05 PM

Participants: Cyrus, Marc, Rooz, and Dr. Super

Introduction

This week we continued our study of Cardano's Method by using it to solve a cubic equation and then finding the remaining two roots. We also reviewed several important algebraic identities and connected them with completing the square and solving quadratic equations.

The Adventure

We began the session with Cyrus solving a cubic equation using Cardano's Formula. After finding one root, he divided the cubic polynomial by $x - x_1$ using long division to obtain a quadratic polynomial. He then solved the quadratic to find the remaining two roots. Marc completed the same activity independently, while I helped Rooz through the steps. Marc had no difficulty, but Rooz needed more guidance.

Since the remaining quadratic was $x^2 - 4x + 4$, I took the opportunity to review the four primary algebraic (geometric) identities:

- $a^2 + b^2 = c^2$ (The Pythagorean Theorem)
- $(a + b)^2$
- $(a - b)^2$
- $a^2 - b^2$

Marc and Cyrus had explored these visual proofs two years earlier, and Rooz had seen them the previous year. Marc demonstrated several of the proofs, and Rooz came up to help. He especially enjoys the visual proofs.

Next, I showed the students how completing the square follows naturally from these algebraic identities and prepared them for three graphing problems on quadratic equations.

Marc and Cyrus followed the presentation well, but Rooz kept reminding me, "You're going too fast!" I smiled because he was keeping up remarkably well. He simply needed a little more time, and I tried to give him both the time and the encouragement to continue.

The students then completed the three graphing exercises and found the roots of each quadratic. Throughout the discussion we compared completing the square with our mean-and-distance method, where a quadratic is written in the form

$$x^2 - 2mx + p,$$

with m representing the mean of the two roots and p their product. The roots are then

$$x_1, x_2 = m \pm d,$$

where

$$d = \sqrt{m^2 - p}.$$

This interpretation emphasizes that $d$is simply the distance of each root from the mean value.

We took our time throughout the session, making sure everyone was comfortable with each method before moving on. I spent extra time with Rooz to help him keep up with the group. Although I think these ideas are challenging for him, he enjoys learning alongside the older boys and continues to work hard.

Overall, it was a very good session, and all three students remained focused throughout the afternoon.

What I Learned

Today's session reinforced the value of moving slowly through difficult material. Solving cubic equations, reviewing the algebraic identities, completing the square, and connecting these ideas to graphing quadratics all fit together naturally. Taking the extra time to help Rooz ensured that everyone finished the session with a solid understanding of the main ideas.

Looking Ahead

Next week we will continue our study of complex numbers and build on these algebraic techniques as we move further into Stephen Welch's series.

Figure 4.1 – Solving a cubic equation using Cardano's Formula.

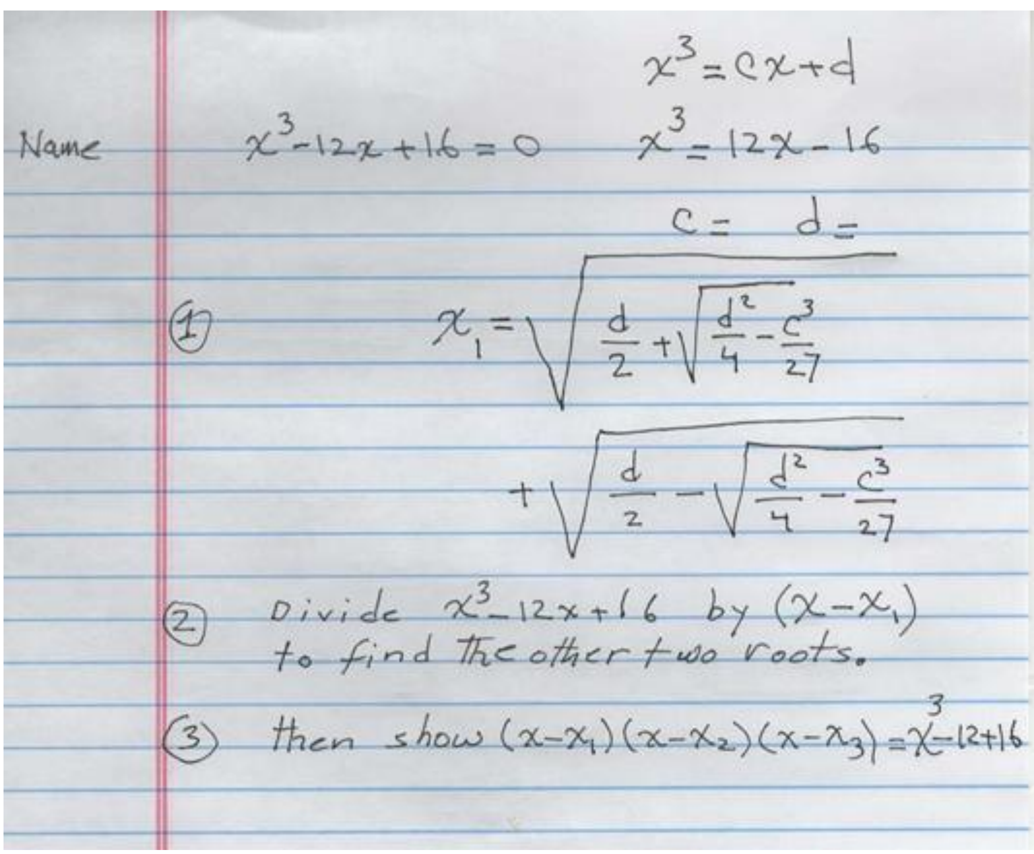


Figure 4.2 – Cyrus's completed solution.

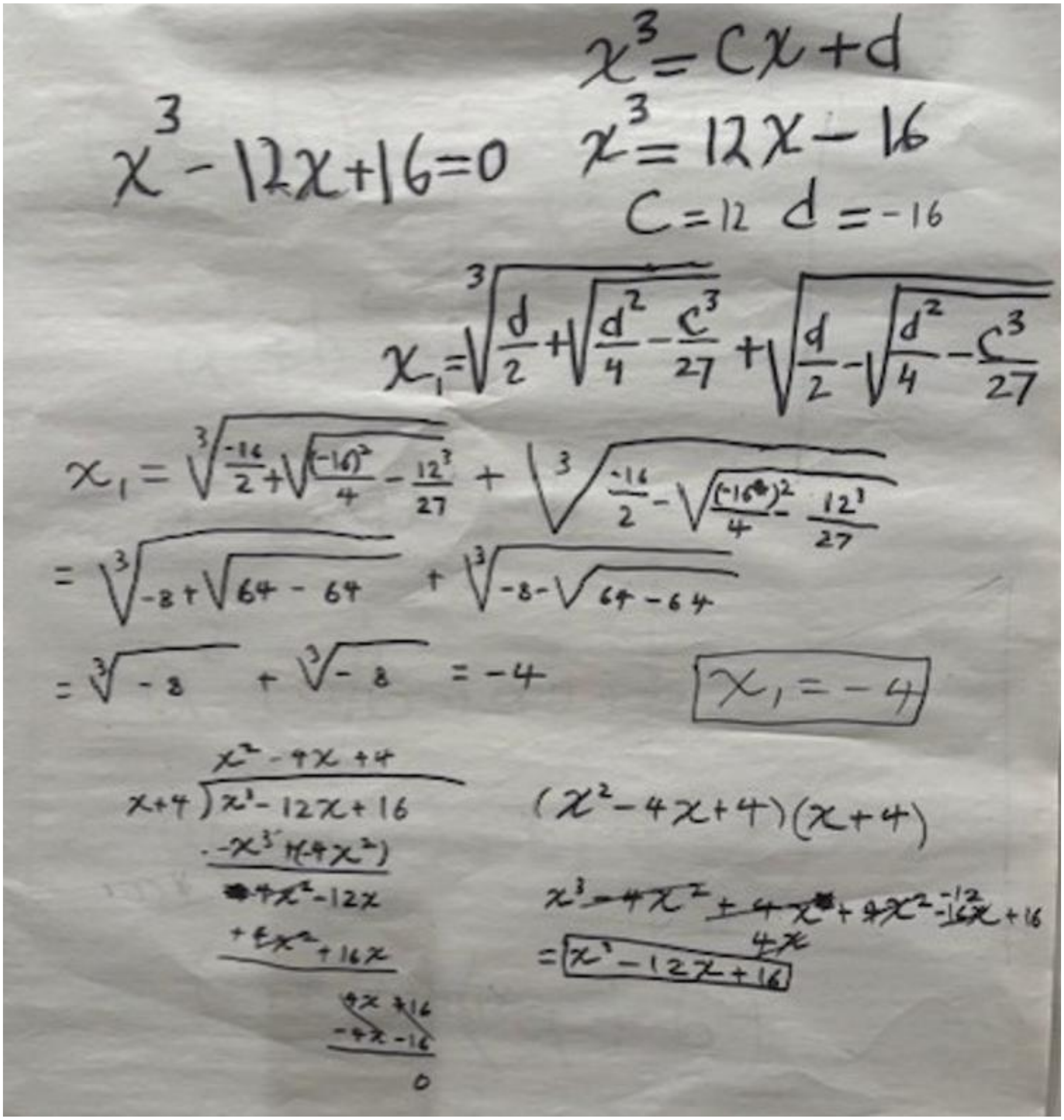
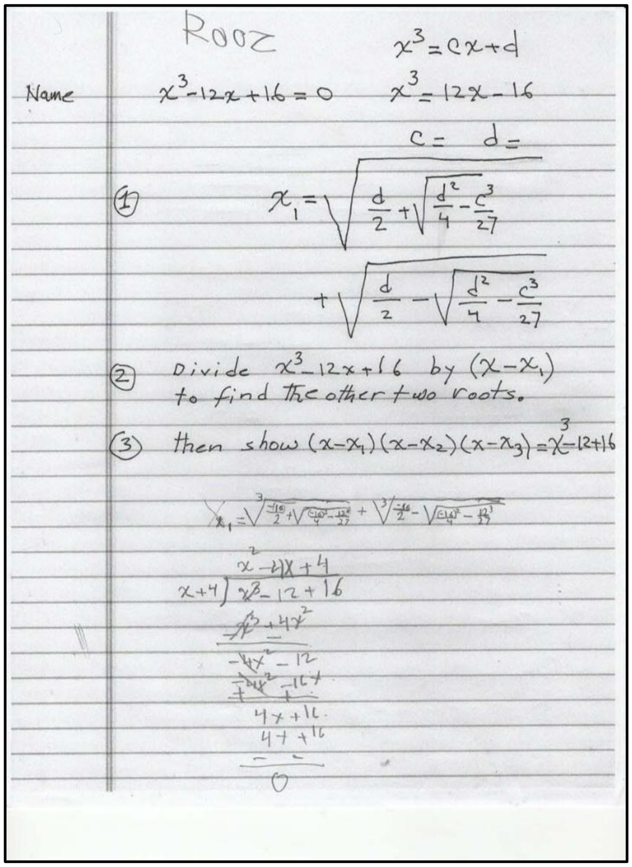


Figure 4.3 – Rooz's completed solution.



Rooz

$x^3 = cx + d$

Name $x^3 - 12x + 16 = 0$ $\quad x^3 = 12x - 16$

$c = \quad d =$

(1) $x_1 = \sqrt{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}} + \sqrt{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^3}{27}}}$

(2) Divide $x^3 - 12x + 16$ by $(x - x_1)$ to find the other two roots.

(3) then show $(x - x_1)(x - x_2)(x - x_3) = x^3 - 12 + 16$

$x_1 = \sqrt[3]{\frac{-16}{2} + \sqrt{\frac{(16)^2}{4} - \frac{12^3}{27}}} + \sqrt[3]{\frac{-16}{2} - \sqrt{\frac{(16)^2}{4} - \frac{12^3}{27}}}$

$x^2 - 4x + 4$

$x + 4 \,)\, x^3 - 12 + 16$

$x^3 + 4x^2$

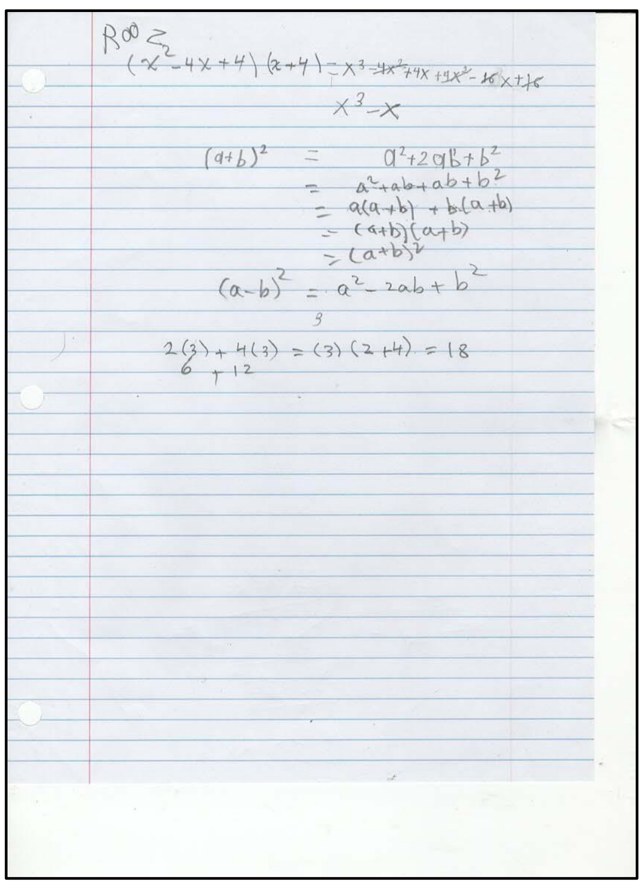
$-4x^2 - 12$

$-4x^2 - 16x$

$4x + 16$

$4x + 16$

0



Rooz 2

$(x^2 - 4x + 4)(x + 4) = x^3 + 4x^2 - 4x^2 + 4x + 4x^2 - 16x + 16$

$x^3 - x$

$(a+b)^2 = a^2 + 2ab + b^2$

$= a^2 + ab + ab + b^2$

$= a(a+b) + b(a+b)$

$= (a+b)(a+b)$

$= (a+b)^2$

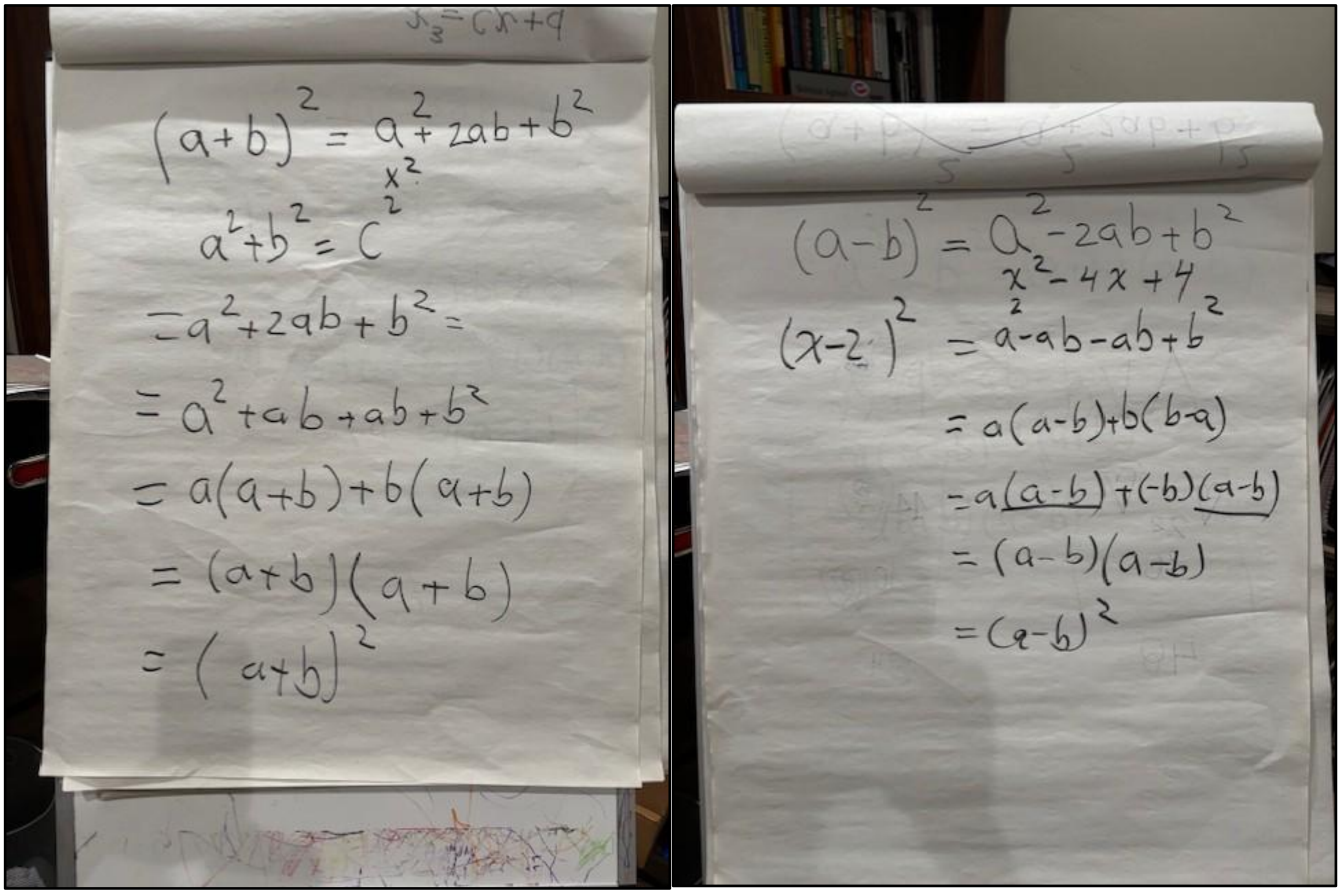
$(a-b)^2 = a^2 - 2ab + b^2$

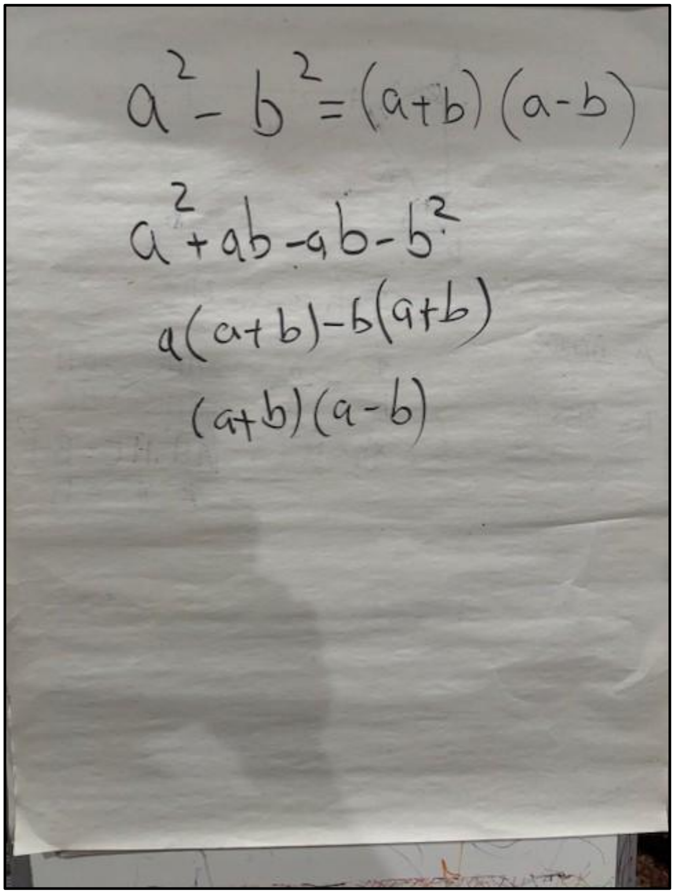
3

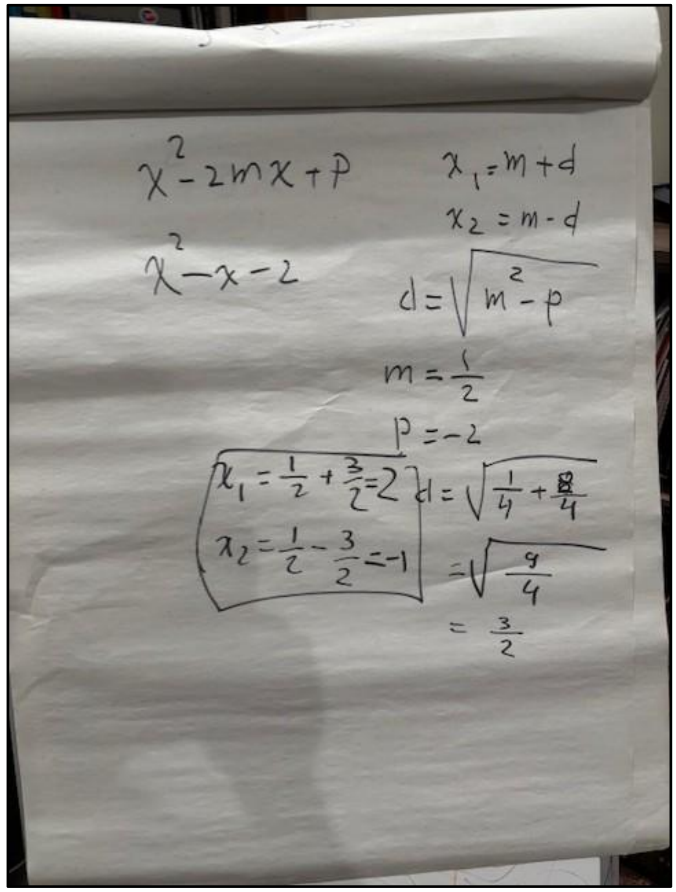
$2(3) + 4(3) = (3)(2+4) = 18$

$6 + 12$

Figure 4.4 – Visual proofs of the algebraic identities.







$a^2 + ab + ab + b^2 = a(a+b) + b(a+b) = (a+b)(a+b)$
$= (a+b)^2$
$a^2 - 2ab + b^2 = a^2 - ab - ab + b^2 = a(a-b) + b(b-a)$
$= a(a-b) + (-b)(a-b) = (a-b)(a-b) = (a-b)^2$

Student Activities and Completed Student Work for Quadratic graphing activities

R002

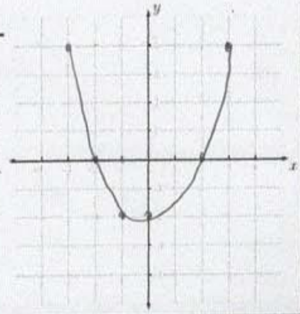
Drill

$x^2 \div 2mx + p$

	Plot	Solve by Factoring (If Possible)	Solve by Quadratic Formula (If Possible)
2.4	$f(x) = x^2 - x - 2$	$0 = x^2 - x - 2$	$0 = x^2 - x - 2$
2.5	$g(x) = x^2 + 3x + 2$	$0 = x^2 + 3x + 2$	$0 = x^2 + 3x + 2$
2.8	$h(x) = x^2 + 1$	$0 = x^2 + 1$	$0 = x^2 + 1$

2.4 $f(x) = x^2 - x - 2$

x	y
-2	4
-1	0
0	-2
1	-2
2	0
3	4



Solve by Factoring:

$0 = x^2 - x - 2$

$x^2 - 2x - 2 \cancel{x} - x - 2$

$= x^2 - 2x + x - 2$

$= x(x-2) + (x-2)$

$= (x-2)(x+1)$

$= 0$

$x - 2 = 0$, $x = 2$

$x + 1 = 0$, $x = -1$

$\boxed{x = 2 \quad x = -1}$

Solve by Quadratic Formula:

$0 = x^2 - x - 2$

$m = \frac{1}{2}$ $p = -2$

$d = \sqrt{m^2 - p}$

$d = \sqrt{\frac{1}{4} + \frac{2}{1}}$

$= \sqrt{\frac{1}{4} + \frac{8}{4}}$

$= \sqrt{\frac{9}{4}}$

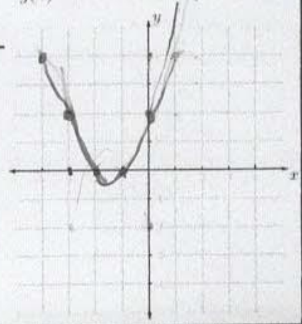
$= \frac{3}{2}$

$x_1 = m + d = 2$

$x_2 = m - d = -1$

2.5 $g(x) = x^2 + 3x + 2$

x	y
-3	2
-2	0
-1	0
0	2
1	6



Solve by Factoring:

$0 = x^2 + 3x + 2$

$0 = x^2 + 2x + (x+2)$

$= x(x+2) + (x+2)$

$= (x+2)(x+1)$

$x = -2$

$x = -1$

Solve by Quadratic Formula:

$0 = x^2 + 3x + 2$

$m = -\frac{3}{2}$ $p = 2$

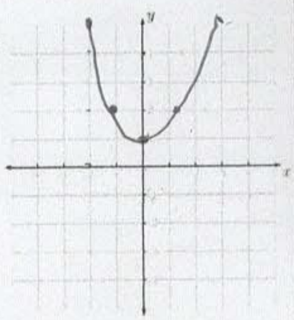
$d = \sqrt{\frac{9}{4} - \frac{8}{4}}$

$= \sqrt{\frac{1}{4}}$

$= \frac{1}{2}$

2.8 $h(x) = x^2 + 1$

x	y
-2	
-1	
0	
1	
2	



Solve by Factoring:

$0 = x^2 + 1$

$0 = x^2 + ix - ix + i$

$= x(x+i) - i(x+i)$

$= (x+i)(x-i)$

Solve by Quadratic Formula:

$0 = x^2 + 1$

$m = 0$

$p = 1$

$d = \sqrt{0 + 1}$

$d = i$

$x_1 = i$

$x_2 = -i$

Cyrus

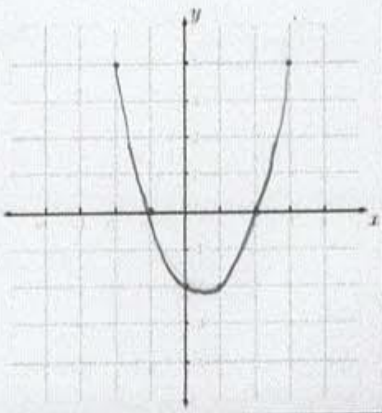
Drill

	Plot	Solve by Factoring (If Possible)	Solve by Quadratic Formula (If Possible)

2.4

$f(x) = x^2 - x - 2$

x	y
-2	4
-1	0
0	-2
1	-2
2	0
3	4



Solve by Factoring: $0 = x^2 - x - 2$

$x^2 - 2x + 2x - x - 2$

$= x^2 - 2x + x - 2$

$= x(x-2) + (x-2)$

$= (x-2)(x+1)$

Solve by Quadratic Formula: $0 = x^2 - x - 2$

$m = \frac{1}{2}$

$p = -2$

$d = \sqrt{\frac{1}{4} + \frac{8}{4}} = \sqrt{\frac{9}{4}}$

$= \frac{3}{2}$

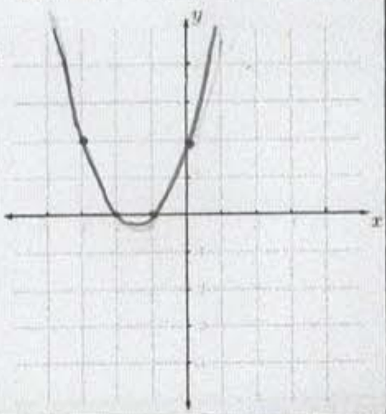
$x_1 = -1$

$x_2 = 2$

2.5

$g(x) = x^2 + 3x + 2$

x	y
-3	2
-2	0
-1	0
0	2
1	6



Solve by Factoring: $0 = x^2 + 3x + 2$

$x^2 + 2x + x + 2$

$= x(x+2) + (x+2)$

$(x+2)(x+1)$

Solve by Quadratic Formula: $0 = x^2 + 3x + 2$

$m = -\frac{3}{2}$

$p = 2$

$d = \sqrt{\frac{9}{4} - \frac{8}{4}} = \frac{1}{2}$

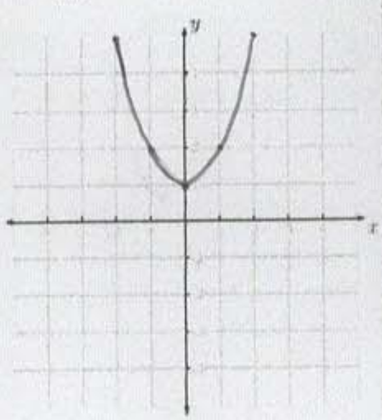
$x_1 = -2$

$x_2 = -1$

2.8

$h(x) = x^2 + 1$

x	y
-2	5
-1	2
0	1
1	2
2	5



Solve by Factoring: $0 = x^2 + 1$

Solve by Quadratic Formula: $0 = x^2 + 1$

$m = 0$

$p = 1$

$d = \sqrt{0^2 - 1} = i$

$x_1 = -i$

$x_2 = i$

C

Adventure 5 — Imaginary Numbers are Real (Videos 10–13)

Date: Wednesday, September 10, 2025

Time: 1:45 PM – 2:45 PM

Participants: Cyrus, Marc, Rooz, and Dr. Super

Introduction

This week we reached the final four videos in Stephen Welch's *Imaginary Numbers are Real* series. Before watching them, however, we began with a fascinating activity from Tadashi Tokieda that introduced the students to the ideas of friction and center of gravity.

The Adventure

We started the session with an activity from **Tadashi Tokieda** showing **how we can balance a ruler using our two fingers.**. I demonstrated the activity and jokingly told the boys that I had trained my fingers to find the center of gravity of a ruler, even when a weight was attached to one end. Cyrus and Marc immediately wanted to try it and did not buy my story! To my surprise, Rooz never actually tried the experiment himself, although at the end he said he wanted to use the long ruler after we had already moved on. Even so, he watched very carefully.

After Cyrus and Marc experimented for a while, I asked them how the two fingers always managed to meet at the center of gravity. Cyrus explained that the heavy side of the ruler tried to move downward while the lighter side moved upward. I then asked what physical phenomenon caused the fingers to move the way they did, and Marc correctly answered **friction**. Putting the two ideas together, just as Tadashi explained, we saw that the finger on the lighter side experiences less friction and therefore moves more easily until both fingers eventually meet at the center of gravity. The students really enjoyed the activity.

I also introduced the students briefly to Tadashi Tokieda's Numberphile videos. His channel contains many beautiful demonstrations of fundamental mathematical and physical ideas, and his personal story is equally inspiring.

Next, I told the students that they were about to watch the final four videos in the *Imaginary Numbers are Real* series. I explained that these videos are considerably more difficult than the earlier ones and admitted that even after watching them more than five times, I still did not feel that I understood them completely.

The four videos took about thirty minutes to watch, and the students were completely captivated. The topics included complex functions and Riemann surfaces, leading to a beautiful visualization of why the equation

$$x^2 + 1 = 0$$

has the two solutions $i$and $-i$. The graphics and presentation were exceptional. I was a little concerned that one of the students—especially Rooz—might stop in the middle and say that he did not understand what was going on. That never happened. Instead, they watched quietly from beginning to end.

I highly recommend watching these videos more than once. My plan is to watch them again with Cyrus and Rooz, and I also recommend that Marc and Hugo watch them together. Breaking the series into two sessions—Videos 1–9 and Videos 10–13—works very well.

Although it was only 2:45, I decided that the students had enough to think about, so I skipped the worksheet I had prepared. Instead, I went to get lunch and some bread and olive oil for Marc. When I returned, I discovered that the boys had already started the additional video on e^x. Cyrus noticed that it would take another twenty-five minutes and decided not to continue. I really had not planned for them to watch this extra video, but all three were once again captivated by the graphics. Even if the mathematics was beyond them, they could still sense the beauty of what was being explained.

I felt this was a wonderful session. More than anything else, it gave the students a glimpse of what higher mathematics looks and feels like.

What I Learned

Sometimes students do not need to understand every detail in order to appreciate beautiful mathematics. The final four videos are challenging, even for adults, but they inspired curiosity and wonder. That alone made the afternoon worthwhile.

Looking Ahead

Our next Adventure will return to these ideas with hands-on activities and discussions. By slowing down and revisiting the material, I hope the students will develop an even deeper appreciation for the remarkable world of complex numbers.

Photographs

Figure 5.1 – [Tadashi Tokieda's balancing ruler activity.]

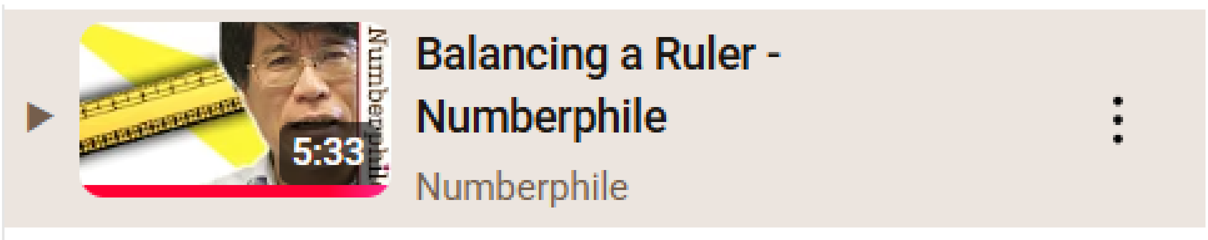


Figure 5.2 [Tadashi's channel in Numberphile].

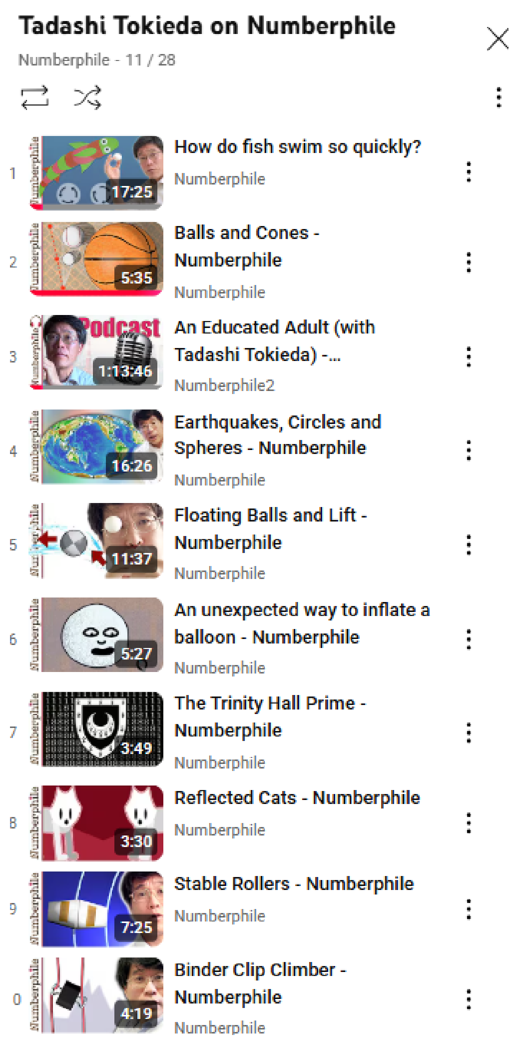
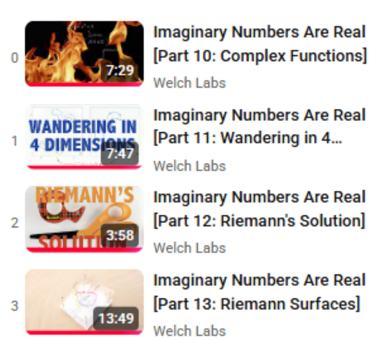


Figure 5.3 – ***Imaginary Numbers are Real*** **Videos 10–13.**



Adventure 6 — Building a Riemann Surface

Date: Wednesday, September 17, 2025

Time: 1:40 PM – 2:40 PM

Participants: Cyrus, Marc, Rooz, and Dr. Super

Introduction

This week the students built their own three-dimensional Riemann Surface. It was a wonderful hands-on activity that brought Stephen Welch's final video to life and gave the students a visual way to understand how the two imaginary roots of the equation

$$z^2 + 1 = 0$$

can be ow displayed.

The Adventure

Today I guided the students in building their own **Riemann Surface** for the equation

$$w = z^2 + 1.$$

We began by cutting out a heavy paper version, which was easier to handle, and then built the final model using transparencies. I helped them assemble and tape the pieces together to complete the three-dimensional construction.

When I first built this model a few days earlier, it took me almost two hours to get everything right. To my surprise, the boys completed their models in less than half an hour! I was impressed by how quickly they understood the construction and how carefully they worked together.

After finishing the models, we watched the final video in Stephen Welch's series once again. This time it made much more sense. I think I finally understood what Stephen was explaining, although I had already watched the complete series at least six times.

The students were excited to see how their Riemann Surfaces illustrated what was happening in four dimensions and how the two imaginary roots, $i$and $-i$, could finally be visualized. It was a wonderful conclusion to an exceptional series. I may return to several of these ideas in future Math Circles by selecting additional activities on complex numbers.

What I Learned

Building the Riemann Surface transformed a difficult mathematical idea into something the students could hold in their hands. Watching the final video after completing the model made the concepts much more meaningful for all of us.

Looking Ahead

Although we have reached the end of Stephen Welch's series, our exploration of complex numbers is not over. We will continue to revisit these ideas and build on them in future Math Circle Adventures.

Figure 6.1 – Cyrus, Marc, and Rooz proudly displaying their completed Riemann Surfaces.

Figure 6.2 – Building the three-dimensional Riemann Surface.

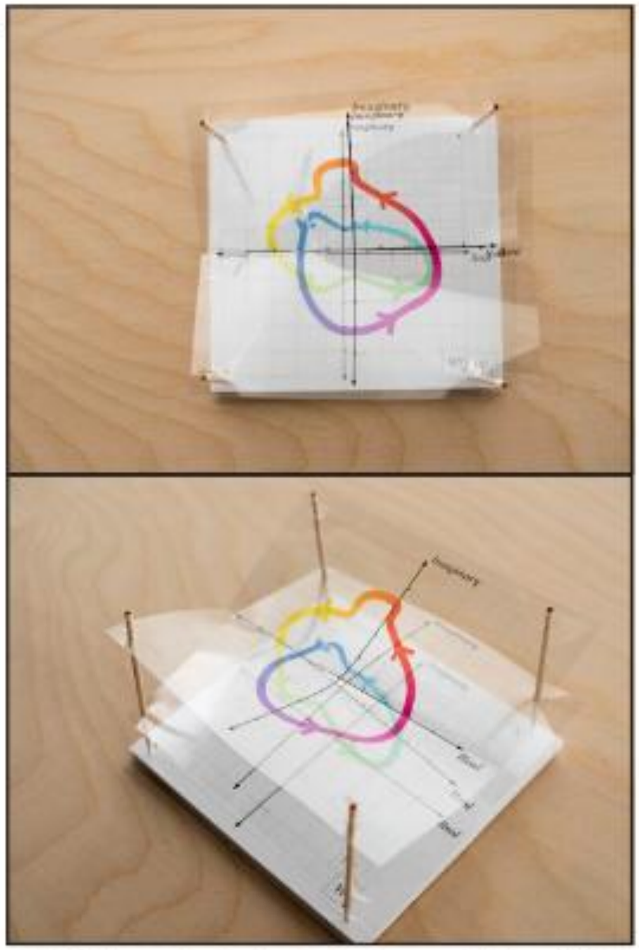


Figure 61 | The Riemann Surface for Our Function $z = f^{-1}(w) \pm \sqrt{w}$. We made this surface by printing the paths from Figure 60 on transparencies, cutting the planes at their discontinuities, and taping the paths together.

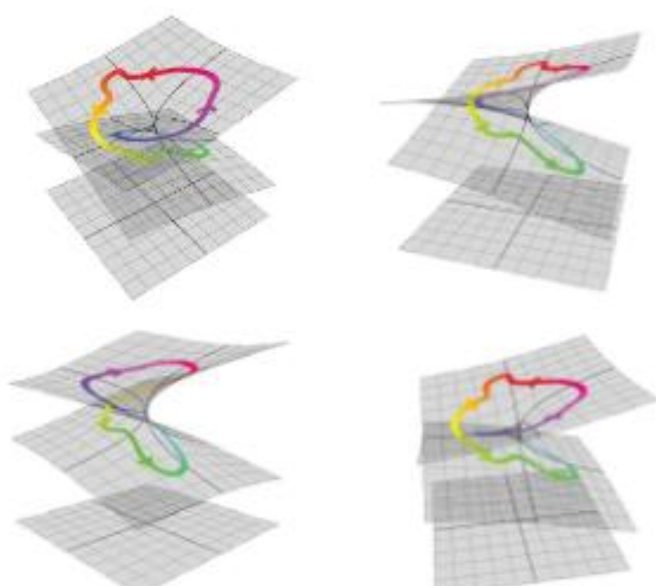
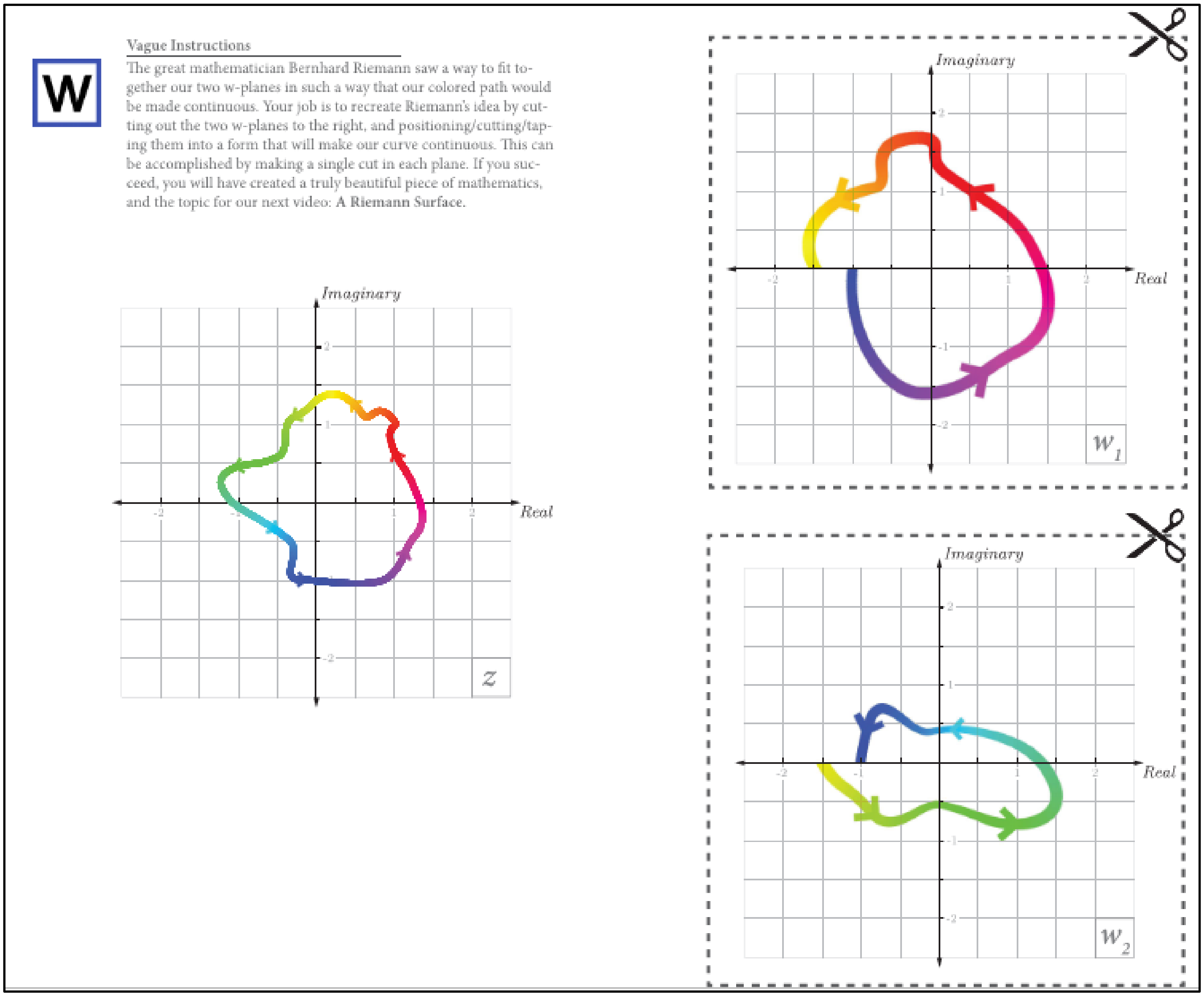


Figure 62 | The Riemann Surface for Our Function $z = f^{-1}(w) \pm \sqrt{w}$. We made this surface using the programming language Python and the plotting library Plotly

Figure 6.3 – The transparency templates used to construct the surface.



Adventure 7 — Euler's Equation

Date: Wednesday, September 24, 2025

Time: 1:40 PM – 3:05 PM

Participants: Cyrus, Marc, Rooz, and Dr. Super

Introduction

This week we watched the final video in the *Imaginary Numbers are Real* series, **The Most Beautiful Equation in Mathematics, Explained Visually (Euler's Formula).** Although the material is challenging, I felt it was worth sharing with the students because it brings together many of the central ideas of mathematics in a beautiful and visual way.

The Adventure

Before watching the video, I had the students complete a worksheet based on the previous lesson on the equation

$$w = z^2 + 1,$$

and compare it with the corresponding real equation. They completed a table to find the roots by squaring complex numbers. Everyone needed a little help, especially with the fact that $(-i)^2 = -1$. Rooz needed more assistance, while Cyrus and Marc corrected a few of their own calculations as they worked through the problem. We also discussed that for a complex number to be a root, both the real and imaginary parts of $w$must be zero, leading once again to the roots $i$and $-i$.

Next, I wanted them to see that the complex equation

$$w = z^2 - 4$$

has the same roots as the real equation

$$y = x^2 - 4.$$

They completed function tables for both equations and graphed the real parabola. We discussed how the corresponding complex function produces a similar surface that still intersects the real axis at $-2$and 2, where the imaginary part is zero. Although this is difficult to see directly from the picture, it can be imagined from the graph.

We then watched **The Most Beautiful Equation in Mathematics, Explained Visually (Euler's Formula)**, stopping several times to discuss the ideas being presented. Afterwards, the students completed another worksheet illustrating Briggs' method. Because I had accidentally misplaced one of the arrows on the worksheet, the calculators were producing incorrect values. Cyrus immediately noticed that the first answer had to be exactly 2, as indicated on the worksheet. After correcting the mistake, they completed the second table for $2^{0.587}$and obtained the correct result.

We did not have time to complete the final worksheet, but we discussed how the number 0.693 is related to Euler's number through the expression

$$2^{1/0.693} \approx e = 2.71828 \ldots$$

I told the students we would review this idea briefly next week. Marc, in particular, was impressed by this relationship.

Although this was a difficult session, the students stayed engaged throughout, and I was impressed by their perseverance.

What I Learned

Euler's Formula brings together many important ideas—powers, logarithms, complex numbers, polar coordinates, and Euler's number. Even though the material is challenging, exposing students to these ideas gives them a glimpse of the beauty and unity of higher mathematics.

Looking Ahead

Our final Adventure will conclude our journey by exploring one last beautiful topic and reflecting on everything we have learned throughout the series.

Figure 7.1 – Visual introduction to Euler's Formula.

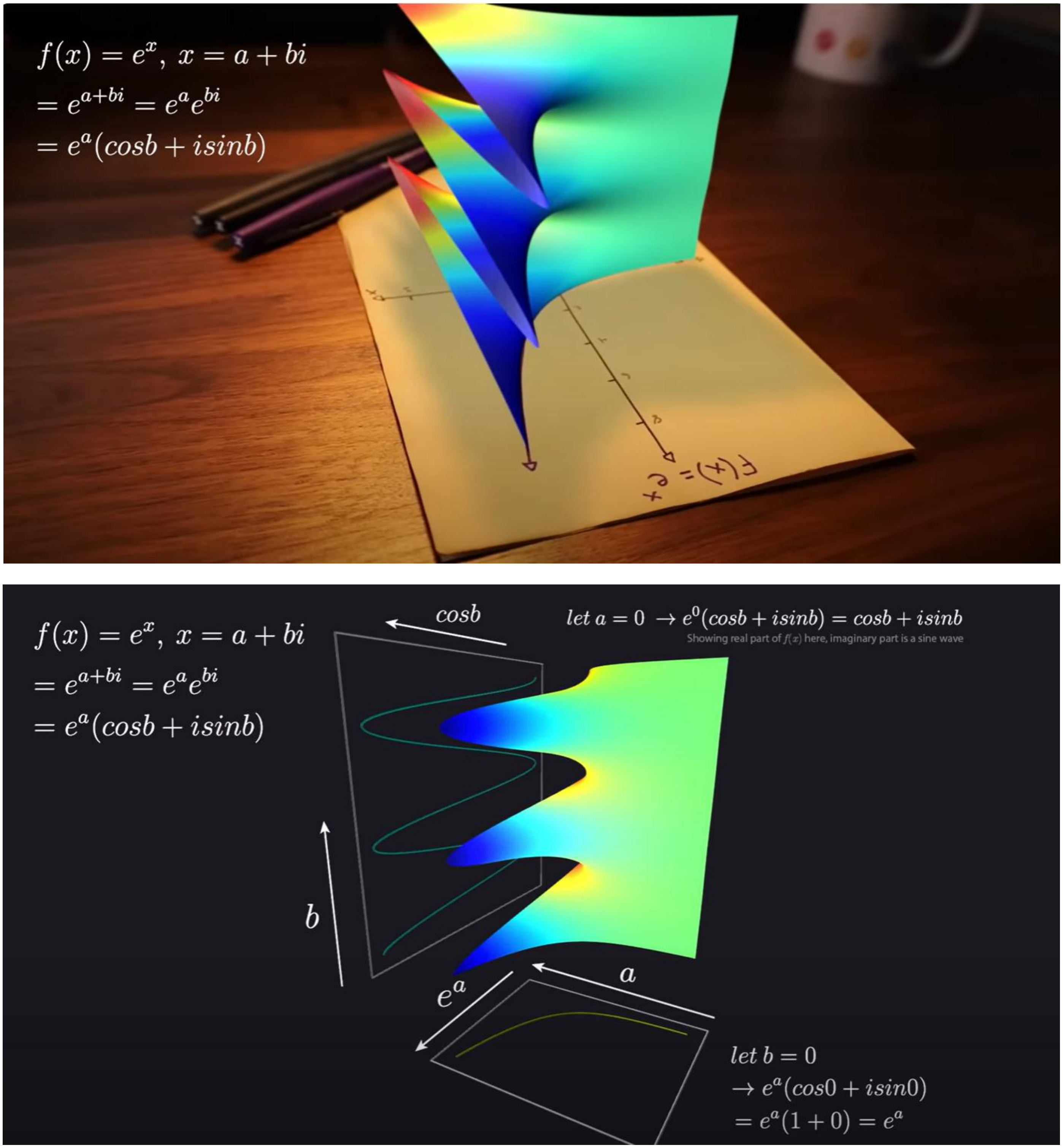


Figure 7.2 – Worksheet on $w = z^2 + 1$.

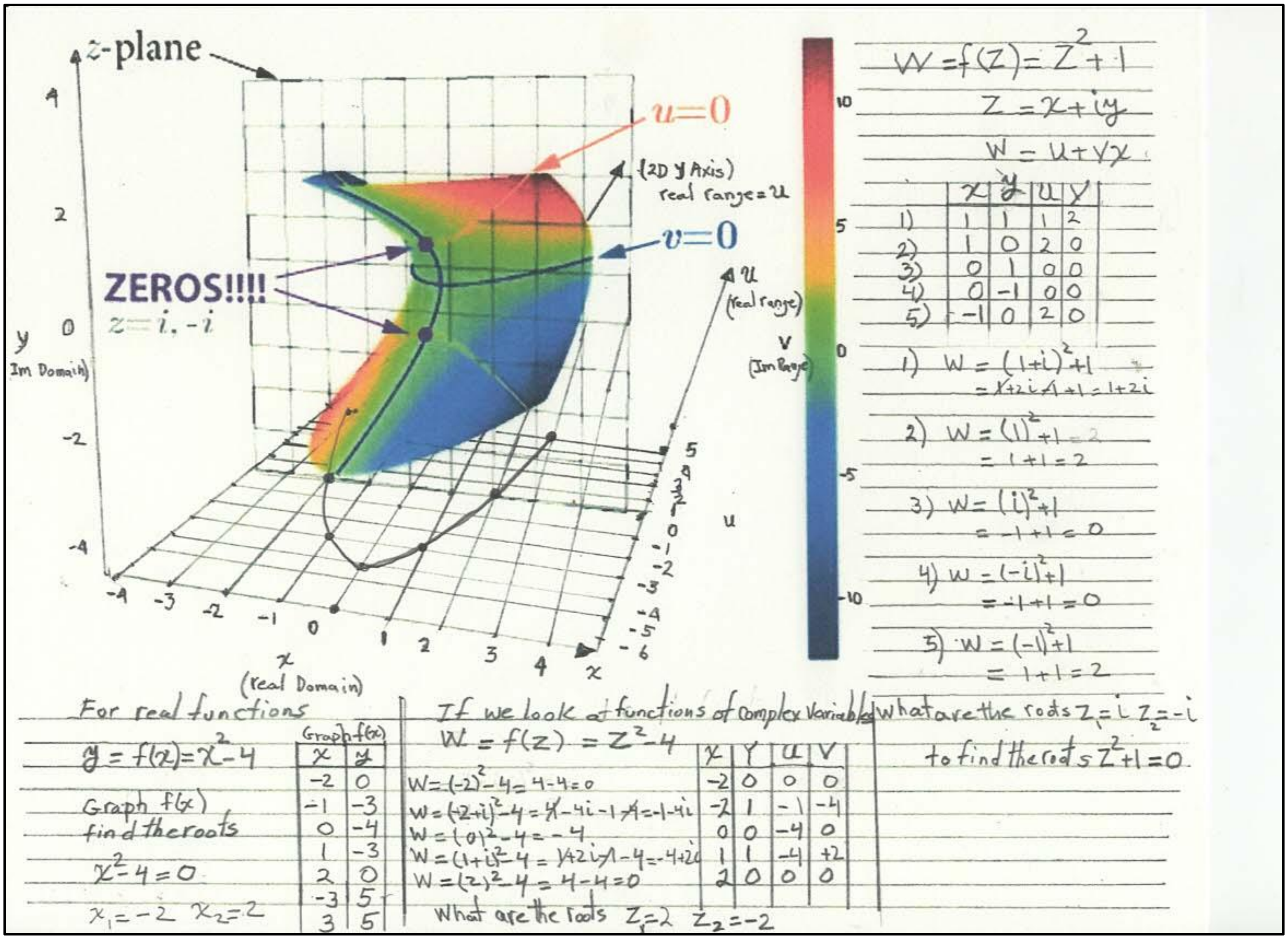


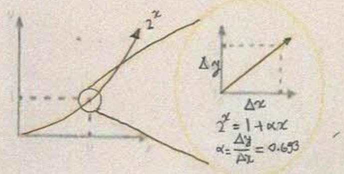
Figure 7.3 – Briggs' method activity.

Calculating 2 to the power 0.587

$$2^x \approx \left(1+0.693\cdot\frac{x}{N}\right)^N, \text{ let } N = 1000$$

$$2^{0.587} \approx \left(1+0.693\cdot\frac{0.587}{1000}\right)^{1000} = 1.000406791^{1000}$$

$$\approx 1.502$$



$2^x = (1+0.693\cdot x/N)^N$

$e^x = \lim_{N\to\infty}(1+x/N)^N$

x	2^x
1	2
1/2	1.414
1/4	1.189
...	1.0013574
$1/2^{20}$	1.000000661

x	2^x
0.587	$2^{0.587}$
...	
0.000587	1.00040696

$2^1 = (1+0.693(\frac{1}{1,048,576}))^{1,048,576}$

2 =

$2^{0.587} = (1+0.693(1.00040696))^{1000}$

$= (1+0.693\times(\frac{0.587}{1000}))^{1000}$

Figure 7.4 – Discovering Euler's Formula worksheet.

Discovering Eulers Formula - Greates Formula in Math!

$2^i \approx \left(1+0.693\cdot\frac{i}{6}\right)^6 = (1.007\angle 6.588°)^6$

$2^i \approx (1.007)^6\angle(6.588°\cdot 6) = 1.041\angle 39.531°$

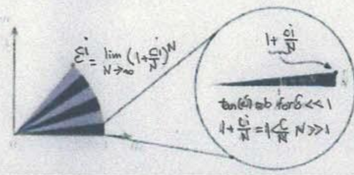
$2^i \approx 0.803+0.662i$ .693 Radians

$e^i \approx (1+i/1000)^{1000}$

$\approx (1.0000005\angle 0.057295°)^{1000}$

$\approx 1.0005\angle 57.296°$ – 1 Radian

$= \cos(1)+i\sin(1) \approx 0.54+0.84i$



$e^{ci} = \lim_{N\to\infty}(1+\frac{ci}{N})^N$

$= \lim_{N\to\infty}(1\angle\frac{c}{N})^N$

$= 1\angle c = \cos(\theta)+i\sin(\theta)$

$\boxed{e^{i\theta} = \cos(\theta)+i\sin(\theta)}$

$2^{1/.693}$ =

No. of Triangles	Radius	Angle
1	1.0000005	0.057295
...		
1000		

57.296 = 1 Radian

$e^{i\theta}=\cos(\theta)+i\sin(\theta)$

Adventure 8 — Euler's Equation and Chaos Theory

Date: Wednesday, October 1, 2025

Time: 1:45 PM – 3:00 PM

Participants: Cyrus, Marc, Rooz, and Dr. Super

Introduction

Our final Adventure began with a review of **The most beautiful equation in math, explained visually [Euler's Formula]** before introducing the students to one of the most fascinating ideas in modern mathematics—**Chaos Theory**. It was a fitting conclusion to a series that began with quadratic equations and ended with one of the great discoveries of the twentieth century.

The Adventure

We began by reviewing the graphics from the previous lesson to make sure the students understood the colorful surface and the hyperbolic shape. We also reviewed why both the real equation

$$y = x^2 - 4$$

and the complex equation

$$w = z^2 - 4$$

have the same two roots, $-2$and 2. We spent about five minutes discussing this, and I think they understood it about as well as I did.

Next, we repeated the worksheet for calculating $2^1$and $2^{0.587}$. I helped Rooz, and all three students needed a little more practice using their calculators. Marc worked a little faster, but I think it was good for everyone to become more comfortable with calculators.

I then reviewed the worksheet leading to Euler's Formula. Rather than having the students complete every step themselves, I explained the details so that we would have enough time for the Chaos Theory activities that I had planned. Next, I gave them the picture of

$$e^x, x = a + bi,$$

and asked them to determine how the colorful $\cos(b)$surface appears when $a = 0$, and how the graph reduces to $e^a$when $b = 0$.

Before watching the video **This equation will change how you see the world (the logistic map),** I introduced the Logistic Map,

$$x_{n+1} = rx_n(1 - x_n),$$

and explained how it models the growth of a population. Using their calculators, the students experimented with several values of r. For $r = 2.3$, they found that the population converged to a fixed point. For $r = 0.5$, the population eventually died out. When they tried $r = 3.2$, they discovered a bifurcation with two repeating values, and for $r = 3.5$, they eventually found four repeating values. Cyrus suggested trying $r = \sqrt{2}$, and it also converged to a fixed point.

We then watched the video, which repeated these experiments and showed how chaotic behavior begins near $r = 3.59$. The video also explained the relationship between the Logistic Map and the Mandelbrot Set, introduced the Feigenbaum Constant (**The Feigenbaum Constant (4.669) - Numberphile**), and showed that many different functions exhibit the same remarkable behavior.

To conclude the afternoon, I told them how, in 1977, a colleague and I had unknowingly encountered chaotic behavior while studying income distribution. At the time, we dismissed it as a computer error. I also gave each student laminated sheets showing the Logistic Map and its relationship to the Mandelbrot Set. Finally, I recommended James Gleick's excellent book *Chaos: Making a New Science* and suggested that they read it in another year or two.

The students really enjoyed the session. They struggled a little with their calculators at first, but by the end they were much more comfortable using them.

What I Learned

Chaos Theory provided a wonderful ending to our Adventures. The students saw how a simple equation could produce remarkably rich and unexpected behavior. More importantly, they experienced another example of how mathematics continues to surprise us, even when we begin with a very simple idea.

Looking Back

Over eight Wednesday afternoons, the students explored ideas that ranged from quadratic equations to imaginary numbers, Cardano's Formula, Riemann Surfaces, Euler's Formula, and finally Chaos Theory. They did not master every topic, nor was that the goal. My hope was simply to give them a glimpse of the beauty of higher mathematics and to inspire them to continue asking questions.

I believe Stephen Welch's *Imaginary Numbers are Real* series accomplished exactly that.